

# Hierarchical Coded Caching

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## Abstract

Caching of popular content during off-peak hours is a strategy to reduce network loads during peak hours. Recent work has shown significant benefits of designing such caching strategies not only to deliver part of the content locally, but also to provide coded multicasting opportunities even among users with different demands. Exploiting both of these gains was shown to be approximately optimal for caching systems with a single layer of caches.

Motivated by practical scenarios, we consider in this work a hierarchical content delivery network with two layers of caches. We propose a new caching scheme that combines two basic approaches. The first approach provides coded multicasting opportunities within each layer; the second approach provides coded multicasting opportunities across multiple layers. By striking the right balance between these two approaches, we show that the proposed scheme achieves the optimal communication rates to within a constant multiplicative and additive gap. We further show that there is no tension between the rates in each of the two layers up to the aforementioned gap. Thus, both layers can simultaneously operate at approximately the minimum rate.

## I. INTRODUCTION

The demand for high-definition video streaming services such as YouTube and Netflix is driving the rapid growth of Internet traffic. In order to mitigate the effect of this increased load on the underlying communication infrastructure, content delivery networks deploy storage memories or caches throughout the network. These caches can be populated with some of the popular content during off-peak traffic hours. This cached content can then be used to reduce the network load during peak traffic hours when users make the most requests.

Content caching has a rich history, see for example [1] and references therein. More recently, it has been studied in the context of video-on-demand systems for which efficient content placement schemes have been proposed in [2], [3] among others. The impact of different content popularities on the caching schemes has been investigated for example in [4]–[6]. A common feature among the caching schemes studied in the literature is that those parts of a requested file that are available at nearby caches are served locally, whereas the remaining file parts are served via orthogonal transmissions from an origin server hosting all the files.

Recently, [7], [8] proposed a new caching approach, called *coded caching*, that exploits cache memories not only to deliver part of the content locally, but also to create coded multicasting opportunities among users with different demands. It is shown there that the reduction in rate due to these coded multicasting opportunities is significant and can be on the order of the number of users in the network. The setting considered in [7], [8] consists of a single layer of caches between the origin server and the end users. The server communicates directly with all the caches via a shared link, and the objective is to minimize the required transmission rate by the server. For this basic network scenario, coded caching is shown there to be optimal within a constant factor. These results have been extended to nonuniform demands in [9] and to online caching systems in [10].

In practice, many caching systems consist of not only one but multiple layers of caches, usually arranged in a tree-like hierarchy with the origin server at the root node and the users connected to the leaf caches [2], [11], [12]. Each parent cache communicates with its children caches in the next layer, and the objective is to minimize the transmission rates in the various layers.

There are several key questions when analyzing such hierarchical caching systems. A first question is to characterize the optimal tradeoff between the cache memory sizes and the rates of the links connecting the layers. One particular point of interest is if there is any tension between the rates in the different layers in the network. In other words, if we reduce the rate in one layer, does it necessarily increase the rate in other layers? If there is no such tension, then both layers can simultaneously operate at minimum rate. A second question is how to extend the coded caching approach to this setting. Can we simply apply the single-layer scheme from [7], [8] in each layer separately or do we need to apply coding across several layers in order to minimize transmission rates?

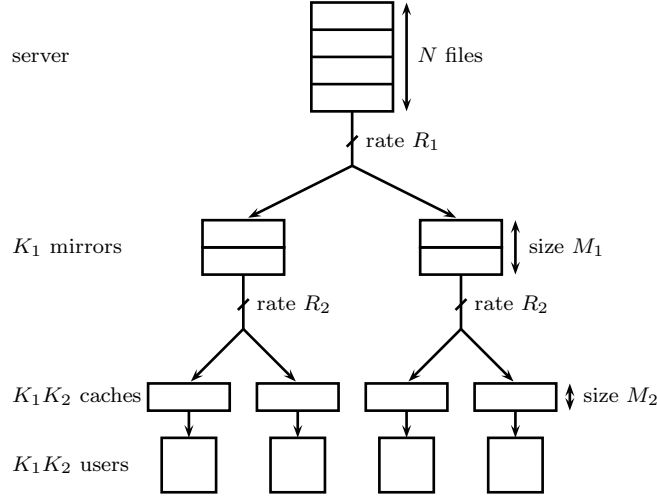


Fig. 1. System setup for the hierarchical caching problem: A server hosting  $N$  files is connected to  $K_1$  mirrors each able to store  $M_1$  of the files. Each of the mirrors, in turn, is connected to  $K_2$  caches each able to store  $M_2$  of the files. A single user is attached to each of these caches. Once the mirrors and caches are filled, each user requests one of the  $N$  files. The aim is to minimize the rate  $R_1$  from the server to the mirrors and the rate  $R_2$  from the mirrors to the caches. In the figure,  $N = 4$ ,  $K_1 = K_2 = 2$ ,  $M_1 = 2$ , and  $M_2 = 1$ .

In this work, we focus on a hierarchical caching system with two layers of caches as depicted in Fig. 1. For simplicity, we will refer to the first layer of caches as mirrors. We propose a new caching scheme exploiting two types of coded caching opportunities: The first type involves only a single layer at a time, i.e., it operates between a node and its direct children. These single-layer coding opportunities are available over the link connecting the origin server to the mirrors and also in the link connecting each mirror to the user caches. The second type involves two layers at a time. These two-layer opportunities are available between the origin server and the user caches. We show that, by striking the right balance between these two types of coded caching opportunities, the proposed caching scheme attains the approximately optimal memory-rate tradeoff to within a constant additive and multiplicative gap. Due to the possible interaction between the two cache layers, the network admits many different prefetching and delivery approaches. It is thus perhaps surprising that a combination of these two basic schemes is sufficient to achieve the approximately optimal memory-rate tradeoff. Furthermore, investigating the achievable rates also reveals that there is no tension between the rates over the first and second layers up to the same aforementioned gap. Thus, both layers can simultaneously operate at approximately minimum rate.

The remainder of the paper is organized as follows. We describe the problem setting in Section II and provide some preliminaries in Section III. Section IV presents our main results and discusses their engineering implications. Section V introduces the proposed caching scheme and characterizes its performance. The proofs of our main results are discussed in Section VI and their details are provided in the appendices. Appendix B proves information-theoretic bounds on the performance of any caching scheme. The proof of the constant multiplicative and additive gap between the performance of the proposed scheme and the optimal caching scheme is provided in Appendices C and D.

## II. PROBLEM SETTING

We consider a hierarchical content delivery network as illustrated in Fig. 1 in Section I. The system consists of a single origin server hosting a collection of  $N$  files each of size  $F$  bits. The server is connected through an error-free broadcast link to  $K_1$  mirror sites, each with memory of size  $M_1 F$  bits. Each mirror, in turn, is connected through an error-free broadcast link to  $K_2$  users. Thus, the system has a total of  $K_1 K_2$  users. Each user has an associated cache memory of size  $M_2 F$  bits. The quantities  $M_1$  and  $M_2$  are the normalized memory sizes of the mirrors and user caches, respectively. We refer to the  $j$ th user attached to mirror  $i$  as “user  $(i, j)$ ” and the corresponding cache as “cache  $(i, j)$ ”. Throughout, we will focus on the most relevant case where the number of files  $N$  is larger than the total number of users  $K_1 K_2$  in the system, i.e.,  $N \geq K_1 K_2$ .

The content delivery system operates in two phases: a *placement phase* followed by a *delivery phase*. The placement phase occurs during a period of low network traffic. In this phase, all the mirrors and user caches store content related to the  $N$  files (possibly using randomized strategies), while satisfying the corresponding memory constraints. Crucially, this is done without any prior knowledge of future user requests. The delivery phase occurs during a period of high network traffic. In this phase, each user requests one of the  $N$  files from the server. Formally, the user requests can be represented as a matrix  $\mathbf{D}$  with entry  $d_{i,j} \in \{1, 2, \dots, N\}$  denoting the request of user  $(i, j)$ . The user requests are forwarded to the corresponding mirrors and further on to the server. Based on the requests and the stored contents of the mirrors and the user caches during the placement phase, the server transmits a message  $X^D$  of size at most  $R_1 F$  bits over the broadcast link to the mirrors. Each mirror  $i$  receives the server message and, using its own memory content, transmits a message  $Y_i^D$  of size at most  $R_2 F$  bits over its broadcast link to users  $(i, 1), (i, 2), \dots, (i, K_2)$ . Using only the contents of its cache  $(i, j)$  and the received message  $Y_i^D$  from mirror  $i$ , each user  $(i, j)$  attempts to reconstruct its requested file  $d_{i,j}$ .

For a given request matrix  $\mathbf{D}$ , we say that the tuple  $(M_1, M_2, R_1, R_2)$  is *feasible for request matrix  $\mathbf{D}$*  if, for large enough file size  $F$ , each user  $(i, j)$  is able to recover its requested file  $d_{i,j}$  with probability<sup>1</sup> arbitrarily close to one. We say that the tuple  $(M_1, M_2, R_1, R_2)$  is *feasible* if it is feasible for all possible request matrices  $\mathbf{D}$ . The object of interest in the remainder of this paper is the feasible rate region:

**Definition.** For memory sizes  $M_1, M_2 \geq 0$ , the *feasible rate region* is defined as

$$\mathcal{R}^*(M_1, M_2) \triangleq \text{closure}\{(R_1, R_2) : (M_1, M_2, R_1, R_2) \text{ is feasible}\}. \quad (1)$$

## III. PRELIMINARIES

The proposed achievable scheme for the hierarchical caching setting makes use of the coded caching scheme developed for networks with a single layer of caches. In this section, we recall this single-layer caching scheme.

Consider the special case of the hierarchical caching setting with no cache memory at the users and only a single user accessing each mirror, i.e.,  $M_2 = 0$  and  $K_2 = 1$ . Let the normalized mirror memory size be  $M_1 = M$  and the number of mirrors  $K_1 = K$ . This results in a system with only a single layer of caches (namely the mirrors).

Note that for this single-layer scenario, each mirror needs to recover the files requested by its corresponding user and then forward the entire file to it. Thus, a transmission rate of  $R_2 = K_2 = 1$  over the link from the mirror to the user is both necessary and sufficient in this case. The goal is to minimize the transmission rate  $R_1$  from the server to the mirrors.

This single-layer setting was recently studied in [7], [8], where the authors proposed a coded caching scheme. For future reference, we recall this scheme in Algorithm 1 and illustrate it below in Example 1. The authors showed that rate  $R_1 = r(M/N, K)$  is feasible in this setting, where  $r(\cdot, \cdot)$  is given by

$$r\left(\frac{M}{N}, K\right) \triangleq \left[ K \cdot \left(1 - \frac{M}{N}\right) \cdot \frac{N}{KM} \left(1 - \left(1 - \frac{M}{N}\right)^K\right) \right]^+ \quad (2)$$

<sup>1</sup>The feasibility of a tuple corresponds to a random variable because of the possible randomization of the placement and delivery phases.

with  $[x]^+ \triangleq \max\{x, 0\}$ . The right hand side of (2) consists of three terms. The first term is the rate without caching. The second term, called *local caching gain*, represents the savings due to a fraction of each file being locally available. The third term, called *global caching gain*, is the gain due to coding. It is shown in [8] that this achievable rate  $R_1$  is within a constant factor of the minimum achievable rate for this single-layer setting for any value of  $N$ ,  $K$ , and  $M$ . We will refer to the placement and delivery procedures of the single-layer coded caching scheme in Algorithm 1 as  $\text{BasePlacement}(N, K, M)$  and  $\text{BaseDelivery}(N, K, M)$ , respectively.

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**Algorithm 1** Single-Layer Coded Caching [8]

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- $[K] \triangleq \{1, 2, \dots, K\}, [N] \triangleq \{1, 2, \dots, N\}$
- Request vector  $\mathbf{d} = (d_1, d_2, \dots, d_K)$
- In Line 9,  $\oplus$  denotes bit-wise XOR operation. For any subset  $\mathcal{S} \subset [K]$  of mirrors,  $V_{j,\mathcal{S}}$  denotes the bits of file  $d_j$  requested by user  $j$  stored exclusively at mirrors in  $\mathcal{S}$ .

1: **procedure** BASEPLACEMENT

2:   **for**  $i \in [K], n \in [N]$  **do**

3:     mirror  $i$  independently stores a subset of  $\frac{MF}{N}$  bits of file  $n$ , chosen uniformly at random

4:   **end for**

5: **end procedure**

6: **procedure** BASEDELIVERY( $\mathbf{d}$ )

7:   **for**  $s = K, K-1, \dots, 1$  **do**

8:     **for**  $\mathcal{S} \subset [K] : |\mathcal{S}| = s$  **do**

9:       server sends  $\oplus_{j \in \mathcal{S}} V_{j,\mathcal{S} \setminus \{j\}}$

10:    **end for**

11:   **end for**

12: **end procedure**

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**Example 1** (*Single-Layer Coded Caching [8]*). Consider the single-layer setting as described above with  $N = 2$  files and  $K = 2$  mirrors each of size  $M_1 = M \in [0, 2]$ . For ease of notation, denote the files by  $A$  and  $B$ . In the placement phase of Algorithm 1, each mirror stores a subset of  $MF/N = MF/2$  bits of each of the two files, chosen uniformly and independently at random. Each bit of a file is thus stored in a given mirror with probability  $M/N = M/2$ .

Consider file  $A$  and notice that we can view it as being composed of  $2^K = 4$  subfiles

$$A = (A_\emptyset, A_1, A_2, A_{1,2}),$$

where  $A_\mathcal{S}$  denotes the bits of file  $A$  which are exclusively stored in the mirrors in  $\mathcal{S}$ . For example,  $A_1$  denotes the bits of file  $A$  which are stored only in mirror 1, and  $A_{1,2}$  denotes the bits of file  $A$  which are available in both mirrors 1 and 2. For large enough file size  $F$ , we have by the law of large numbers that for any subset  $\mathcal{S}$ ,

$$|A_\mathcal{S}| \approx \left(\frac{M}{2}\right)^{|\mathcal{S}|} \left(1 - \frac{M}{2}\right)^{2-|\mathcal{S}|} F.$$

File  $B$  can similarly be partitioned into subfiles.

In the delivery phase, suppose for example that the first user requests file  $A$  and the second user requests file  $B$ . By Line 9 in Algorithm 1, the server transmits  $A_2 \oplus B_1$ ,  $A_\emptyset$ , and  $B_\emptyset$  where  $\oplus$  denotes bit-wise XOR.

Consider mirror 1 whose corresponding user has requested file  $A$ . Mirror 1 already knows the subfiles  $A_1, A_{1,2}$  from its cache memory. Further, the server's transmission provides the subfile  $A_\emptyset$ . Finally, from

$A_2 \oplus B_1$  transmitted by the server, the mirror can recover  $A_2$  since it has  $B_1$  stored in its cache memory. Thus, from the contents of its memory and the server transmission, mirror 1 can recover  $A = (A_\phi, A_1, A_2, A_{1,2},)$  and then forward it to its attached user. Similarly, mirror 2 can recover file  $B$  and forward it to its attached user. The number of bits transmitted by the server is given by

$$\frac{M}{2} \left(1 - \frac{M}{2}\right) F + 2 \left(1 - \frac{M}{2}\right)^2 F = 2 \cdot \left(1 - \frac{M}{2}\right) \cdot \frac{2}{2M} \left(1 - \left(1 - \frac{M}{2}\right)^2\right) F.$$

which agrees with the expression in (2).  $\diamond$

While the above discussion focuses on  $K_2 = 1$  user accessing each mirror, the achievable scheme can easily be extended to  $K_2 > 1$  by performing the delivery phase in  $K_2$  stages with one unique user per mirror active in each stage. The resulting rate over the first link is [8, Section V]

$$R_1 = K_2 \cdot r(M_1/N, K_1). \quad (3)$$

#### IV. MAIN RESULTS

As the main result of this paper, we provide an approximation of the feasible rate region  $\mathcal{R}^*(M_1, M_2)$  for the general hierarchical caching problem with two layers. We start by introducing some notation. For  $\alpha, \beta \in [0, 1]$ , define the rates

$$R_1(\alpha, \beta) \triangleq \alpha K_2 \cdot r\left(\frac{M_1}{\alpha N}, K_1\right) + (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1 K_2\right), \quad (4a)$$

$$R_2(\alpha, \beta) \triangleq \alpha \cdot r\left(\frac{\beta M_2}{\alpha N}, K_2\right) + (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2\right), \quad (4b)$$

where  $r(\cdot, \cdot)$  is defined in (2) in Section III. Next, consider the following region:

**Definition.** For memory sizes  $M_1, M_2 \geq 0$ , define

$$\mathcal{R}_C(M_1, M_2) \triangleq \{(R_1(\alpha, \beta), R_2(\alpha, \beta)) : \alpha, \beta \in [0, 1]\} + \mathbb{R}_+^2, \quad (5)$$

where  $\mathbb{R}_+^2$  denotes the positive quadrant,  $R_1(\alpha, \beta), R_2(\alpha, \beta)$  are defined in (4), and the addition corresponds to the Minkowski sum between sets.

As will be discussed in more detail later, the region  $\mathcal{R}_C(M_1, M_2)$  is the rate region achieved by appropriately sharing the available memory between two basic achievable schemes during the placement phase and then using each scheme to recover a certain fraction of the requested files during the delivery phase. Each of these two schemes is responsible for one of the two terms in  $R_1(\alpha, \beta)$  and  $R_2(\alpha, \beta)$ . The parameters  $\alpha$  and  $\beta$  dictate what fraction of each file and what fraction of the memory is allocated to each of these two schemes. The region  $\mathcal{R}_C(M_1, M_2)$  is thus the rate region achieved by all possible choices of the parameters  $\alpha$  and  $\beta$ .

Our main result shows that, for any memory sizes  $M_1, M_2$ , the region  $\mathcal{R}_C(M_1, M_2)$  just defined approximates the feasible rate region  $\mathcal{R}^*(M_1, M_2)$ .

**Theorem 1.** Consider the hierarchical caching problem in Fig. 1 with  $N$  files,  $K_1$  mirrors, and  $K_2$  users accessing each mirror. Each mirror and user cache has a normalized memory size of  $M_1$  and  $M_2$ , respectively. Then we have

$$\mathcal{R}_C(M_1, M_2) \subseteq \mathcal{R}^*(M_1, M_2) \subseteq c_1 \cdot \mathcal{R}_C(M_1, M_2) - c_2,$$

where  $\mathcal{R}^*(M_1, M_2)$  and  $\mathcal{R}_C(M_1, M_2)$  are defined in (1) and (5), respectively, and where  $c_1$  and  $c_2$  are finite positive constants independent of all the problem parameters.

Theorem 1 shows that the region  $\mathcal{R}_C(M_1, M_2)$  is indeed feasible (since  $\mathcal{R}_C(M_1, M_2) \subseteq \mathcal{R}^*(M_1, M_2)$ ). Moreover, the theorem shows that, up to a constant additive and multiplicative gap, the scheme achieving  $\mathcal{R}_C(M_1, M_2)$  is optimal (since  $\mathcal{R}^*(M_1, M_2) \subseteq c_1 \cdot \mathcal{R}_C(M_1, M_2) - c_2$ ).

The proof of Theorem 1 is presented in Section VI. The proof actually shows a slightly stronger result than stated in the theorem. Recall that the parameters  $\alpha$  and  $\beta$  control the weights of the split between the two simple coded caching schemes mentioned above. In general, one would expect a tension between the rates  $R_1(\alpha, \beta)$  and  $R_2(\alpha, \beta)$  over the first and second hops of the network. In other words, the choice of  $\alpha$  and  $\beta$  minimizing the rate  $R_1(\alpha, \beta)$  over the first hop will in general *not* minimize the rate  $R_2(\alpha, \beta)$  over the second hop.

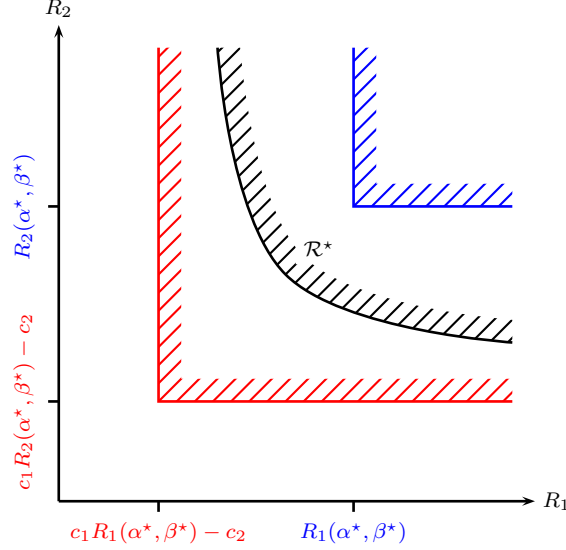


Fig. 2. For fixed memory values  $M_1$  and  $M_2$ , the figure qualitatively depicts the feasible rate region  $\mathcal{R}^*$  and its bounds. As shown in the figure, the feasible rate region  $\mathcal{R}^*$  can be bounded by two rectangular regions with corner points  $(R_1(\alpha^*, \beta^*), R_2(\alpha^*, \beta^*))$  and  $(c_1 R_1(\alpha^*, \beta^*) - c_2, c_1 R_2(\alpha^*, \beta^*) - c_2)$ . Thus, up to the constant additive and multiplicative gap, there is no tension between the rates  $R_1$  over the first hop and the rate  $R_2$  over the second hop of the optimal scheme for the hierarchical caching problem.

However, the proof of Theorem 1 shows that there exists  $\alpha^*$  and  $\beta^*$  (depending on  $N$ ,  $M_1$ ,  $M_2$ ,  $K_1$ , and  $K_2$ ) such that  $R_1(\alpha^*, \beta^*)$  and  $R_2(\alpha^*, \beta^*)$  are *simultaneously* approximately minimized. Thus, surprisingly, there is in fact *no tension* between the rates over the first hop and the second hop for the optimal hierarchical caching scheme up to a constant additive and multiplicative gap (see Fig. 2). The next example shows that this is by no means obvious, and, indeed, we conjecture that it is likely only true approximately.

**Example 2.** Consider a setting with a single mirror  $K_1 = 1$  and memory sizes  $M_1 = M_2 = N/2$ . Assume we use the proposed caching scheme with parameters  $\alpha = 1/2$  and  $\beta = 0$ . As we will see later, this corresponds to placing one half of each file at the mirror and the other half at each of the caches. By (4), we see that the rate tuple  $(R_1, R_2) = (0, K_2/2)$  is achievable. Clearly, this minimizes the rate  $R_1$  over the first hop. However, it is far from optimal for the second hop.

Now, assume we use the proposed caching scheme with parameters  $\alpha = \beta = 1/2$ . By (4), this achieves the rate tuple  $(R_1, R_2) \approx (1/2, 1)$ . Observe that for an increase in rate of  $1/2$  over the first link, we were able to decrease the rate of the second link from  $K/2$  to just one.

We conjecture that the rate tuple  $(R_1, R_2) = (0, 1)$  itself is not achievable.<sup>2</sup> If true, this implies that

<sup>2</sup>This is because in order to achieve rate 0 over the first link, the mirror and each user together must store the entire content, which suggests that the cached contents of the mirror and each user do not overlap. However, to achieve rate 1 over the second link, the mirror needs to be able to exploit coded multicasting opportunities between the users, which suggests that the cached contents of the mirror and the users should overlap. This tension suggests that the rate tuple  $(0, 1)$  is not achievable.

there is tension between the two rates but that this tension accounts for at most a constant additive and multiplicative gap.  $\diamond$

Before we provide the specific values of  $\alpha^*$  and  $\beta^*$ , we describe the two schemes controlled by these parameters in slightly more detail. Both schemes make use of the coded caching scheme for networks with a single layer of caches from [7], [8] as recalled in Section III.

The first scheme uses a very natural decode-and-forward type approach. It uses the single-layer scheme between the server and the  $K_1$  mirrors. Each mirror decodes all messages for its children and re-encodes them using the single-layer scheme between the mirror and its  $K_2$  attached users. Thus, this first scheme creates and exploits coded multicasting opportunities between the server and the mirrors and between each mirror and its users. The second scheme simply ignores the content of the mirrors and applies the single-layer scheme directly between the server and the  $K_1 K_2$  users. Thus, this second scheme creates and exploits coded multicasting opportunities between the server and all the users. With a choice of  $(\alpha, \beta) = (1, 1)$ , all weight is placed on the first scheme and the second scheme is not used. With a choice of  $(\alpha, \beta) = (0, 0)$ , all weight is placed on the second scheme and the first scheme is not used.

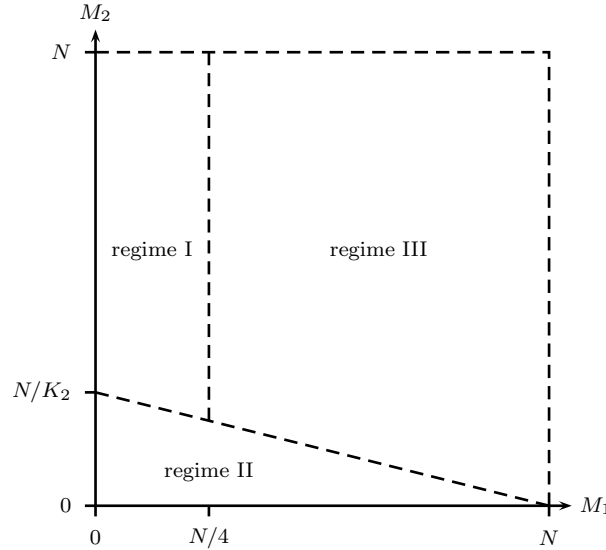


Fig. 3. Different regimes for  $\alpha^*$  and  $\beta^*$ .

With this in mind, let us return to the choice of  $\alpha^*$  and  $\beta^*$ . We consider three different regimes of  $M_1$  and  $M_2$  as depicted in Fig. 3. We set

$$(\alpha^*, \beta^*) = \begin{cases} \left( \frac{M_1}{N}, \frac{M_1}{N} \right) & \text{in regime I,} \\ \left( \frac{M_1}{M_1 + M_2 K_2}, 0 \right) & \text{in regime II,} \\ \left( \frac{M_1}{N}, \frac{1}{4} \right) & \text{in regime III.} \end{cases} \quad (6)$$

Substituting this choice into (4), the corresponding achievable rates are

$$R_1(\alpha^*, \beta^*) \approx \begin{cases} \min \left\{ K_1 K_2, \frac{N}{M_2} \right\} & \text{in regime I,} \\ \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\} & \text{in regime II,} \\ \frac{(N - M_1)^2}{N M_2} & \text{in regime III,} \end{cases} \quad (7a)$$

and

$$R_2(\alpha^*, \beta^*) \approx \min \left\{ K_2, \frac{N}{M_2} \right\}, \quad (7b)$$

where the approximation is up to a constant additive and multiplicative gap as before.

From (6), we see that in every regime we need to share between the two simple schemes. In particular, using the natural decode-and-forward type approach (i.e., scheme one) alone can be highly suboptimal as the next two examples show.

**Example 3.** Let  $M_1 = 0$  and  $M_2 = N$  so that the mirrors have zero memory and the user caches are able to store the entire database of files. This setting falls into regime I. We focus on the rate over the first link from the server to the mirrors. We know that in this example the optimal rate  $R_1$  is 0. By (7a), the rate  $R_1(\alpha^*, \beta^*)$  is approximately equal to 1 (a constant). On the other hand, the rate achieved by using only the first (decode-and-forward) scheme is equal to  $R_1(1, 1) = K_1 K_2$ , which could be much larger.  $\diamond$

**Example 4.** Let  $M_1 = N - N^{2/3}$ ,  $M_2 = N^{1/4}$ ,  $K_1 = 1$ , and  $K_2 = N^{5/6}$ . This setting falls into regime III. By (7a), the rate  $R_1(\alpha^*, \beta^*)$  is approximately equal to  $N^{1/12}$ . On the other hand, the rate achieved by using only the first (decode-and-forward) scheme is approximately equal to  $N^{1/2}$ , which could again be much larger.  $\diamond$

## V. CACHING SCHEMES

In this section, we introduce a class of caching schemes for the hierarchical caching problem. We begin in Sections V-A and V-B by using the BasePlacement and BaseDelivery procedures defined in Section III for networks with a single layer of caches to construct two simple caching schemes for networks with with two layers of caches. We will see in Section V-C how to combine these two schemes to yield a near-optimal scheme for the hierarchical caching problem.

### A. Caching Scheme A

Informally, this scheme places content in the mirrors so that using the server transmission and their own content, each mirror can recover all the files requested by their attached users. In turn, each mirror then acts as a server for these files. Content is stored in the attached user caches so that by using the mirror transmission and their cache content, each user can recover its requested file. See Fig. 4 for an illustration of the scheme.

More formally, in the placement phase, we use the BasePlacement( $N, K_1, M_1$ ) procedure recalled in Section III to store portions of the files  $1, 2, \dots, N$  across the  $K_1$  mirrors. Also, for each mirror  $i$ , we use the BasePlacement( $N, K_2, M_2$ ) procedure to independently store portions of the files  $1, 2, \dots, N$  across caches  $(i, 1), (i, 2), \dots, (i, K_2)$  corresponding to the users with access to mirror  $i$ . In other words, each mirror independently stores a random  $M_1 F/N$ -bit subset of every file, and each user cache independently stores a random  $M_2 F/N$ -bit subset of every file.

During the delivery phase, the server uses the BaseDelivery( $N, K_1, M_1$ ) procedure to the mirrors in order to enable them to recover the  $K_2$  files  $d_{i,1}, d_{i,2}, \dots, d_{i,K_2}$ . In other words, each mirror decodes all



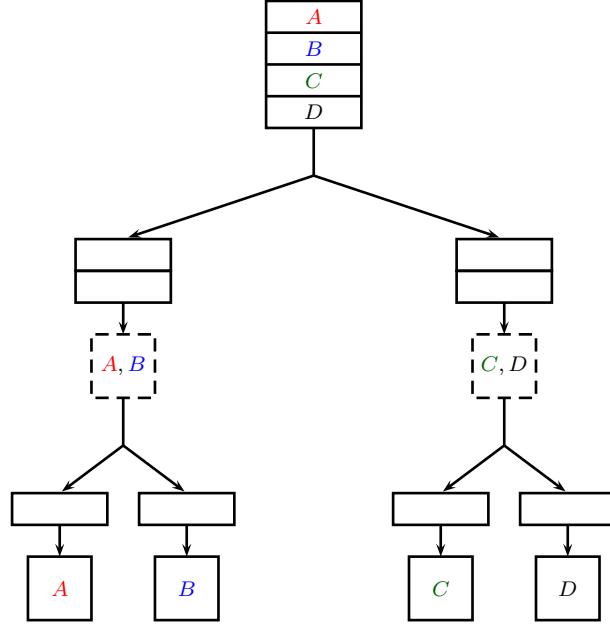


Fig. 4. Caching scheme A for a system with  $K_1 = 2$  mirrors and  $K_2 = 2$  users per mirror. Scheme A uses a decode-and-forward type approach to apply the single-layer coded caching scheme recalled in Section III to a network with two layers. We independently cache content in each of the layers during the placement phase. In the delivery phase, the mirrors decode all the files requested by their users and re-encode them their children. For example, in the figure mirror 1 decodes files  $A, B$  and re-encodes them for the two attached users.

files requested by its attached users. Next, each mirror  $i$  uses the  $\text{BaseDelivery}(N, K_2, M_2)$  procedure to re-encode these files for its  $K_1$  users. This enables each user  $(i, j)$  to recover its requested file  $d_{i,j}$ . Thus, scheme A exploits coded multicasting opportunities between the server and the mirrors and between the mirrors and their users.

The rates for caching scheme A are as follows. By (3), the rate over the link from the server to the mirror is

$$R_1^A \triangleq K_2 \cdot r\left(\frac{M_1}{N}, K_1\right). \quad (8a)$$

By (2), the rate over the link from the mirrors to their users is

$$R_2^A \triangleq r\left(\frac{M_2}{N}, K_2\right). \quad (8b)$$

**Example 5.** Consider the setup in Fig. 4 with  $N = 4$  files,  $K_1 = 2$  mirrors, and  $K_2 = 2$  users per mirror. The mirror and user cache memory sizes are  $M_1 = 2$  and  $M_2 = 1$ , respectively. For ease of notation, denote the files by  $A, B, C$  and  $D$ . Using scheme A, each mirror independently stores a random  $F/2$ -bit subset of every file, and each user cache independently stores a random  $F/4$ -bit subset of every file.

In the delivery phase, assume the four users request files  $A, B, C$ , and  $D$ , respectively. The server uses the  $\text{BaseDelivery}$  procedure to enable the first mirror to recover files  $A$  and  $B$  and to enable the second mirror to recover files  $C$  and  $D$ . This uses a rate of

$$R_1^A = 2 \cdot r(1/2, 2).$$

Mirror 1 then uses the  $\text{BaseDelivery}$  procedure to re-encode the files  $A$  and  $B$  for its attached users. Similarly, mirror 2 uses the  $\text{BaseDelivery}$  procedure to re-encode the files  $C$  and  $D$  for its attached users. This uses a rate of

$$R_2^A = r(1/4, 2).$$

## B. Caching Scheme B

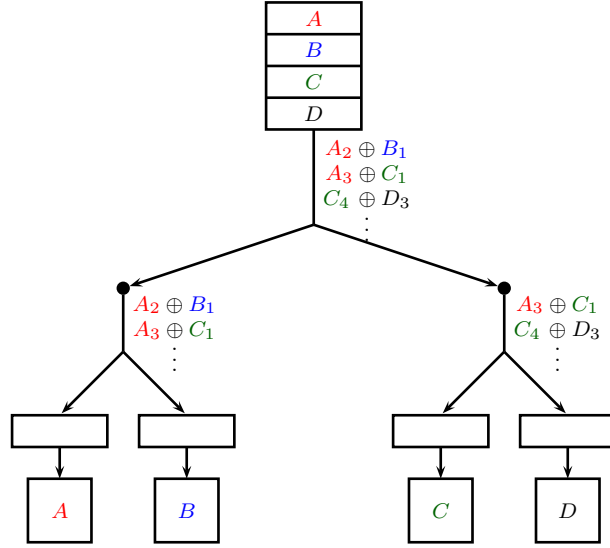


Fig. 5. Caching scheme B for a system with  $K_1 = 2$  mirrors and  $K_2 = 2$  users per mirror. Scheme B ignores the memory at the mirrors and uses the single-layer coded caching scheme recalled in Section III directly between the server and the users. The mirrors are only used to forward the relevant messages transmitted by the server to their users.

Informally, this scheme places content across the  $K_1 K_2$  user caches so that using the server transmissions and its own cache content, each user can recover its requested file. The storage capabilities of the mirrors in the network are completely ignored and the mirrors are only used to forward relevant parts of the server transmissions to the corresponding users. See Fig. 5 for an illustration.

More formally, in the placement phase, we use the  $\text{BasePlacement}(N, K_1 K_2, M_2)$  procedure to store portions of the files  $1, 2, \dots, N$  across the  $K_1 K_2$  user caches and leave all the mirrors empty. In other words, each user cache independently stores a random  $M_2 F/N$ -bit subset of every file.

During the delivery phase, the server uses the  $\text{BaseDelivery}(N, K_1 K_2, M_2)$  procedure directly for the  $K_1 K_2$  users. Recall from the description in Section III that the  $\text{BaseDelivery}$  procedure transmits several sums of file parts. The transmission of mirror  $i$  consists of all those sums transmitted by the server that involve at least one of the  $K_2$  files  $d_{i,1}, d_{i,2}, \dots, d_{i,K_2}$ , requested by its attached users  $(i, 1), (i, 2), \dots, (i, K_2)$ . From the information forwarded by the mirrors, each user is able to recover its requested file. Thus, scheme B exploits coded multicasting opportunities directly between the server and the users across two layers.

The rates for caching scheme B are as follows. By (2), the rate over the link from the server to the mirrors is

$$R_1^B \triangleq r\left(\frac{M_2}{N}, K_1 K_2\right). \quad (9a)$$

Forwarding only the relevant server transmissions is shown in [8, Section V.A] to result in a rate

$$R_2^B \triangleq r\left(\frac{M_2}{N}, K_2\right). \quad (9b)$$

between each mirror and its attached users.

**Example 6.** Consider the setup in Fig. 5 with  $N = 4$  files  $K_1 = 2$  mirrors, and  $K_2 = 2$  users per mirror. The user cache memory size is  $M_2 = 1$  (the mirror memory size  $M_1$  is irrelevant here). For ease of notation, denote the files by  $A, B, C$  and  $D$ . Furthermore, it will be convenient in the remainder of this

example to label the users and caches as 1, 2, 3, 4 as opposed to (1, 1), (1, 2), (2, 1), (2, 2). Using scheme B, each user cache independently stores a random  $F/4$ -bit subset of every file.

In the delivery phase, assume the four users request files  $A$ ,  $B$ ,  $C$ , and  $D$ , respectively. The server uses the BaseDelivery procedure to enable the users to recover their requested files as follows. Consider file  $A$ , and denote by  $A_S$  the bits of file  $A$  stored exclusively at the user caches in  $S \subset \{1, 2, 3, 4\}$ . The transmission from the server to the mirrors is then

$$\begin{aligned} & A_{2,3,4} \oplus B_{1,3,4} \oplus C_{1,2,4} \oplus D_{1,2,3} \\ & A_{2,3} \oplus B_{1,3} \oplus C_{1,2}, \quad A_{2,4} \oplus B_{1,4} \oplus D_{1,2}, \quad A_{3,4} \oplus C_{1,4} \oplus D_{1,3}, \quad B_{3,4} \oplus C_{2,4} \oplus D_{2,3} \\ & A_2 \oplus B_1, \quad A_3 \oplus C_1, \quad A_4 \oplus D_1, \quad B_3 \oplus C_2, \quad B_4 \oplus D_2, \quad C_4 \oplus D_3 \\ & A_\emptyset, \quad B_\emptyset, \quad C_\emptyset, \quad D_\emptyset. \end{aligned}$$

For large enough file size  $F$ , this uses a normalized rate of

$$R_1^B = r(1/4, 4).$$

Let us focus on mirror 1. Since its attached users request files  $A$  and  $B$ , it forwards every sum including parts of either of those files. Thus, mirror 1 transmits

$$\begin{aligned} & A_{2,3,4} \oplus B_{1,3,4} \oplus C_{1,2,4} \oplus D_{1,2,3} \\ & A_{2,3} \oplus B_{1,3} \oplus C_{1,2}, \quad A_{2,4} \oplus B_{1,4} \oplus D_{1,2}, \quad A_{3,4} \oplus C_{1,4} \oplus D_{1,3}, \quad B_{3,4} \oplus C_{2,4} \oplus D_{2,3} \\ & A_2 \oplus B_1, \quad A_3 \oplus C_1, \quad A_4 \oplus D_1, \quad B_3 \oplus C_2, \quad B_4 \oplus D_2, \\ & A_\emptyset, \quad B_\emptyset, \end{aligned}$$

This uses a normalized rate of

$$R_2^B = r(1/4, 2).$$

◇

### C. Generalized Caching Scheme

The generalized scheme divides the system into two subsystems, the first one operated according to caching scheme A and the second one according to caching scheme B. Fix parameters  $\alpha, \beta \in [0, 1]$ . The first subsystem includes the entire memory of each mirror and a  $\beta$  fraction of each user cache memory. The second subsystem includes the remaining  $(1 - \beta)$  fraction of each user cache memory. We split each file into two parts of size  $\alpha F$  and  $(1 - \alpha)F$  bits, respectively. We use scheme A from Section V-A to store and deliver the first parts of the files. Similarly, we use scheme B from Section V-B for the second parts of the files. See Fig. 6 for an illustration.

Since our system is a composition of two disjoint subsystems, the net rate over each transmission link is the sum of the corresponding rates in the two subsystems. From (8), the rates  $R_1^1, R_2^1$  required by scheme A over the first subsystem are

$$R_1^1 = \alpha K_2 \cdot r\left(\frac{M_1}{\alpha N}, K_1\right), \quad (10a)$$

$$R_2^1 = \alpha \cdot r\left(\frac{\beta M_2}{\alpha N}, K_2\right). \quad (10b)$$

Similarly, from (9), the rates  $R_1^2, R_2^2$  required by scheme B over the second subsystem are

$$R_1^2 = (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1 K_2\right), \quad (11a)$$

$$R_2^2 = (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2\right). \quad (11b)$$

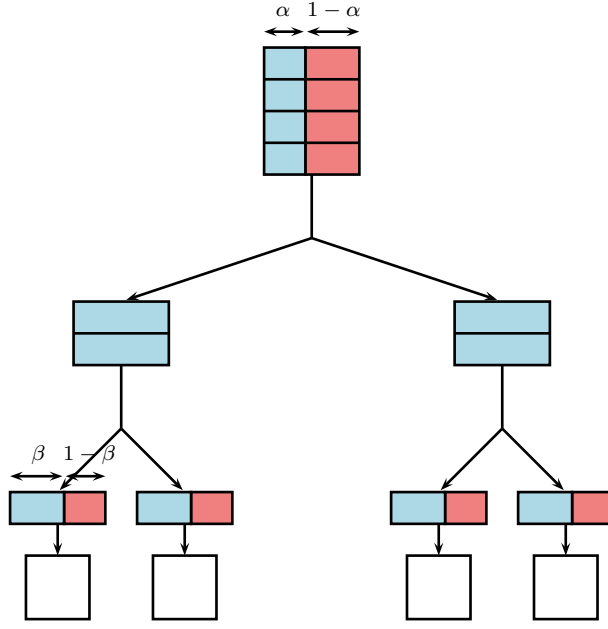


Fig. 6. Generalized caching scheme for a system with  $K_1 = 2$  mirrors and  $K_2 = 2$  users per mirror. For given  $\alpha$  and  $\beta$ , the system can be split into two disjoint subsystems. We use caching scheme A for delivering the first parts of the files over the first subsystem and use caching scheme B for delivering the second parts of the files over the second subsystem.

The formal derivation for these rate expressions is provided in Appendix A.

Combining (10) and (11), the net rates  $R_1 = R_1(\alpha, \beta)$  and  $R_2 = R_2(\alpha, \beta)$  of the generalized caching scheme are

$$R_1(\alpha, \beta) \triangleq R_1^1 + R_1^2 = \alpha K_2 \cdot r\left(\frac{M_1}{\alpha N}, K_1\right) + (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1 K_2\right), \quad (12a)$$

$$R_2(\alpha, \beta) \triangleq R_2^1 + R_2^2 = \alpha \cdot r\left(\frac{\beta M_2}{\alpha N}, K_2\right) + (1 - \alpha) \cdot r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2\right). \quad (12b)$$

Note that this coincides with (4).

#### D. Choice of $\alpha^*$ and $\beta^*$

The generalized caching scheme described in the last section is parametrized by  $\alpha$  and  $\beta$ . We now choose particular values  $\alpha^*$  and  $\beta^*$  for these parameters. Recall from Section IV the three regimes for the memory sizes  $M_1$  and  $M_2$ :

I)  $M_1 + M_2 K_2 \geq N$  and  $0 \leq M_1 \leq N/4$

II)  $M_1 + M_2 K_2 < N$

III)  $M_1 + M_2 K_2 \geq N$  and  $N/4 < M_1 \leq N$

We set

$$(\alpha^*, \beta^*) = \begin{cases} \left(\frac{M_1}{N}, \frac{M_1}{N}\right) & \text{in regime I,} \\ \left(\frac{M_1}{M_1 + M_2 K_2}, 0\right) & \text{in regime II,} \\ \left(\frac{M_1}{N}, \frac{1}{4}\right) & \text{in regime III.} \end{cases} \quad (13)$$

See also (6).

Our proof of Theorem 1 will demonstrate that for any given  $M_1, M_2$ , and this choice of parameters  $\alpha^*, \beta^*$ , the rates  $R_1(\alpha^*, \beta^*), R_2(\alpha^*, \beta^*)$  for the generalized caching scheme are within a constant multiplicative and additive gap of the minimum feasible rates. Before proceeding with the proof of this fact, we provide intuition for the choice of these parameters as well as their impact on the achievable scheme in each of these regimes.

- I) We want to optimize the values of  $\alpha, \beta$  with respect to both  $R_1(\alpha, \beta), R_2(\alpha, \beta)$ . Let us start with the rate  $R_2(\alpha, \beta)$ . From (8b) and (9b), both caching schemes A and B achieve rate  $R_2 = r(M_2/N, K_2)$  on the link from a mirror to its attached users. As we will see later, this rate is in fact approximately optimal for this link. The generalized caching scheme combines caching schemes A and B, and it can be easily verified from (12b) that  $\alpha = \beta$  results in  $R_2(\alpha, \beta) = r(M_2/N, K_2)$ . Thus,  $\alpha = \beta$  is near optimal with respect to  $R_2(\alpha, \beta)$ . To find the optimal common value, we analyze how the rate  $R_1(\alpha, \alpha)$  varies with  $\alpha$  and find that among all values in  $[0, 1]$ , the choice  $\alpha = M_1/N$  results in the near-optimal rate for this regime. Thus, we choose  $(\alpha^*, \beta^*) = (M_1/N, M_1/N)$  for this regime. We now discuss the impact of this choice on the structure of the generalized caching scheme in this regime. Recall that caching scheme A is used to store and deliver the first parts of the files, each of size  $\alpha^*F$  bits. Since  $\alpha^* = M_1/N$  and the mirror memory size is  $M_1F$  bits, this implies that the *entire* first parts of all the  $N$  files can be stored in each mirror. Thus, in this regime, the server does not communicate with the mirrors regarding the first file parts and each mirror, in turn, acts as a sever for these files parts to its attached users. Thus, the generalized caching scheme only exploits coded multicasting opportunities between each mirror and its attached users via caching scheme A and between the server and all the users via caching scheme B.
- II) Observe that the user cache memory  $M_2$  is small in this regime, in particular  $M_2 < N/K_2$ . It can be verified that the rate  $R_2$  in this case has to be at least on the order of  $K_2$ . On the other hand, it is easy to see from (12b) that  $R_2(\alpha, \beta) \leq K_2$  for any choice of parameters  $\alpha, \beta$ . Thus, we only need to optimize  $\alpha, \beta$  with respect to the rate  $R_1(\alpha, \beta)$  over the second link. The optimizing values can be found as  $(\alpha^*, \beta^*) = (M_1/(M_1 + M_2K_2), 0)$ . Recall from Section V-C that caching scheme A is assigned a  $\beta^*$  fraction of each user cache memory. Since  $\beta^* = 0$  for this regime, no user cache memory is assigned for scheme A. Thus, in this regime, the generalized caching scheme only exploits coded multicasting opportunities between the server and its attached mirrors via caching scheme A and between the server and all the users via caching scheme B.
- III) We would like to again choose  $\alpha = \beta = M_1/N$  in this regime as in regime I. However, since the rate  $R_1(\alpha, \beta)$  over the first link increases with  $\beta$ , and since  $M_1/N$  is large (on the order of 1) in this regime, this choice would lead to an unacceptably large value of  $R_1$ . Thresholding  $\beta$  at  $1/4$  (or any other constant for that matter) in this regime enables us to simultaneously achieve the dual purpose of containing its impact on the rate  $R_1$ , while still managing to reduce the rate  $R_2$  sufficiently. Thus, for this regime we choose  $(\alpha^*, \beta^*) = (M_1/N, 1/4)$ . As was the case in regime I, since  $\alpha^* = M_1/N$ , each mirror is able to store the entire first parts of the  $N$  files and thus, the server does not communicate with the mirrors under caching scheme A. Thus, in this regime, the generalized caching scheme only exploits coded multicasting opportunities between each mirror and its attached users via caching scheme A and between the server and all the users via caching scheme B.

#### E. Achievable rates $R_1(\alpha^*, \beta^*), R_2(\alpha^*, \beta^*)$

We next calculate the achievable rates  $R_1(\alpha^*, \beta^*)$  and  $R_2(\alpha^*, \beta^*)$  of the generalized caching scheme describe in Section V-C with the choice of parameters  $\alpha^*$  and  $\beta^*$  as described in Section V-D.

The achievable rates  $R_1(\alpha, \beta), R_2(\alpha, \beta)$  for the generalized caching scheme are given in terms of the

function  $r(\cdot, \cdot)$ , defined in (2). It is easy to see that

$$r\left(\frac{M}{N}, K\right) \leq \begin{cases} \min\left\{K, \frac{N}{M} - 1\right\} & \text{for } M/N \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

As defined in (13), our choice of parameters  $(\alpha^*, \beta^*)$  takes different values for the three different regimes of  $M_1, M_2$ . We evaluate the achievable rates for each of these regimes.

I)  $M_1 + M_2 K_2 \geq N$  and  $0 \leq M_1 \leq N/4$ . Recall from (13) that  $(\alpha^*, \beta^*) = (\frac{M_1}{N}, \frac{M_1}{N})$  in regime I. From (12) and (14), the achievable rates  $R_1(\alpha^*, \beta^*)$  and  $R_2(\alpha^*, \beta^*)$  are upper bounded as

$$\begin{aligned} R_1(\alpha^*, \beta^*) &= \frac{M_1 K_2}{N} \cdot r(1, K_1) + \left(1 - \frac{M_1}{N}\right) \cdot r\left(\frac{M_2}{N}, K_1 K_2\right) \\ &\leq 0 + \min\left\{K_1 K_2, \frac{N}{M_2}\right\} \\ &= \min\left\{K_1 K_2, \frac{N}{M_2}\right\}, \end{aligned} \quad (15a)$$

and

$$\begin{aligned} R_2(\alpha^*, \beta^*) &= \frac{M_1}{N} \cdot r\left(\frac{M_2}{N}, K_2\right) + \left(1 - \frac{M_1}{N}\right) \cdot r\left(\frac{M_2}{N}, K_2\right) \\ &= r\left(\frac{M_2}{N}, K_2\right) \\ &\leq \min\left\{K_2, \frac{N}{M_2}\right\}. \end{aligned} \quad (15b)$$

II)  $M_1 + M_2 K_2 < N$ . Recall from (13) that  $(\alpha^*, \beta^*) = (\frac{M_1}{M_1 + M_2 K_2}, 0)$  in regime II. From (12) and (14), the achievable rate  $R_1(\alpha^*, \beta^*)$  is upper bounded as

$$\begin{aligned} R_1(\alpha^*, \beta^*) &= \frac{M_1 K_2}{M_1 + M_2 K_2} \cdot r\left(\frac{M_1 + M_2 K_2}{N}, K_1\right) + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot r\left(\frac{M_1 + M_2 K_2}{N K_2}, K_1 K_2\right) \\ &\leq \frac{M_1 K_2}{M_1 + M_2 K_2} \cdot \min\left\{K_1, \frac{N}{M_1 + M_2 K_2} - 1\right\} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \min\left\{K_1 K_2, \frac{N K_2}{M_1 + M_2 K_2} - 1\right\} \\ &\leq \frac{M_1}{M_1 + M_2 K_2} \cdot \min\left\{K_1 K_2, \frac{(N - M_1) K_2}{M_1 + M_2 K_2}\right\} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \min\left\{K_1 K_2, \frac{N K_2 - M_1}{M_1 + M_2 K_2}\right\} \\ &\leq \min\left\{K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2}\right\}. \end{aligned} \quad (16a)$$

For the first inequality we have used that  $M_1 + M_2 K_2 < N$  implies  $M_1 < \alpha^* N$  and  $(1 - \beta^*) M_2 = M_2 < (1 - \alpha^*) N$  in the bound (14). On the other hand, from (12) and (14) the achievable rate  $R_2(\alpha^*, \beta^*)$  is trivially upper bounded as

$$R_2(\alpha^*, \beta^*) \leq K_2 = \min\left\{K_2, \frac{N}{M_2}\right\} \quad (16b)$$

where the last equality follows since  $M_2 K_2 < N$  in regime II.

III)  $M_1 + M_2 K_2 \geq N$  and  $N/4 < M_1 \leq N$ . Recall from (13) that  $(\alpha^*, \beta^*) = (\frac{M_1}{N}, \frac{1}{4})$  in regime III. From (12) and (14), the achievable rates  $R_1(\alpha^*, \beta^*)$  and  $R_2(\alpha^*, \beta^*)$  are upper bounded as

$$\begin{aligned} R_1(\alpha^*, \beta^*) &= \frac{M_1 K_2}{N} \cdot r(1, K_1) + \left(1 - \frac{M_1}{N}\right) \cdot r\left(\frac{(1 - 1/4)M_2}{(1 - M_1/N)N}, K_1 K_2\right) \\ &\leq 0 + \left(1 - \frac{M_1}{N}\right) \cdot \min\left\{K_1 K_2, \frac{4(N - M_1)}{3M_2} - 1\right\} \\ &\leq \frac{4(N - M_1)^2}{3NM_2} \end{aligned} \quad (17a)$$

and

$$\begin{aligned} R_2(\alpha^*, \beta^*) &= \frac{M_1}{N} \cdot r\left(\frac{M_2}{4M_1}, K_2\right) + \left(1 - \frac{M_1}{N}\right) \cdot r\left(\frac{3M_2}{4(N - M_1)}, K_2\right) \\ &\leq \frac{M_1}{N} \cdot \min\left\{K_2, \frac{4M_1}{M_2}\right\} + \left(1 - \frac{M_1}{N}\right) \cdot \min\left\{K_2, \frac{4(N - M_1)}{3M_2}\right\} \\ &\leq \frac{M_1}{N} \cdot \min\left\{K_2, \frac{4N}{M_2}\right\} + \left(1 - \frac{M_1}{N}\right) \cdot \min\left\{K_2, \frac{4N}{M_2}\right\} \\ &\leq 4 \min\left\{K_2, \frac{N}{M_2}\right\}. \end{aligned} \quad (17b)$$

Combining (15a), (16a), and (17a), we obtain the following upper bound on the achievable rate  $R_1(\alpha^*, \beta^*)$ :

$$R_1(\alpha^*, \beta^*) \leq \begin{cases} \min\left\{K_1 K_2, \frac{N}{M_2}\right\} & \text{in regime I,} \\ \min\left\{K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1)K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{NK_2 - M_1}{M_1 + M_2 K_2}\right\} & \text{in regime II,} \\ \frac{4(N - M_1)^2}{3NM_2} & \text{in regime III.} \end{cases} \quad (18a)$$

Similarly, combining (15b), (16b), and (17b), we obtain the following upper bound on the achievable rate  $R_2(\alpha^*, \beta^*)$ :

$$R_2(\alpha^*, \beta^*) \leq 4 \min\left\{K_2, \frac{N}{M_2}\right\}. \quad (18b)$$

These upper bounds will be used in the next sections to prove that the achievable rates for our generalized caching scheme are within a constant multiplicative and additive gap of the corresponding lower bounds.

## VI. PROOF OF THEOREM 1

### A. Proof of $\mathcal{R}_C(M_1, M_2) \subseteq \mathcal{R}^*(M_1, M_2)$

Recall the definitions of the feasible rate region  $\mathcal{R}^*(M_1, M_2)$  in (1) and of the region  $\mathcal{R}_C(M_1, M_2)$  in (5) respectively. The result then follows immediately from (12) in Section V-C, which shows that any rate pair in  $\mathcal{R}_C(M_1, M_2)$  is achievable using the generalized caching scheme. ■

B. *Proof of  $\mathcal{R}^*(M_1, M_2) \subseteq c_1 \cdot \mathcal{R}_C(M_1, M_2) - c_2$*

The proof consists of two steps. We first prove lower bounds  $R_1^{\text{lb}}(M_1, M_2)$ ,  $R_2^{\text{lb}}(M_1, M_2)$  on the feasible rates, i.e., for any  $M_1, M_2$ , and  $(R_1, R_2) \in \mathcal{R}^*(M_1, M_2)$ , we have

$$\begin{aligned} R_1 &\geq R_1^{\text{lb}}(M_1, M_2), \\ R_2 &\geq R_2^{\text{lb}}(M_1, M_2). \end{aligned} \quad (19)$$

We compute these lower bounds  $R_1^{\text{lb}}(M_1, M_2)$ ,  $R_2^{\text{lb}}(M_1, M_2)$  in Appendix B.

Next, we show that for any  $M_1, M_2$ , the gap between the achievable rates  $R_1(\alpha^*, \beta^*)$ ,  $R_2(\alpha^*, \beta^*)$  and the lower bounds  $R_1^{\text{lb}}(M_1, M_2)$ ,  $R_2^{\text{lb}}(M_1, M_2)$  is bounded, i.e.,

$$\begin{aligned} R_1^{\text{lb}}(M_1, M_2) &\geq c_1 R_1(\alpha^*, \beta^*) - c_2, \\ R_2^{\text{lb}}(M_1, M_2) &\geq c_1 R_2(\alpha^*, \beta^*) - c_2, \end{aligned} \quad (20)$$

where  $c_1, c_2$  are finite positive constants independent of all the problem parameters. The proof of the above inequalities, bounding the gap between the achievable rates and the lower bounds, involves separate analysis for several different regimes of  $M_1, M_2$ , and is deferred to Appendices C and D.

Combining (19), (20), for any  $M_1, M_2$ , and  $(R_1, R_2) \in \mathcal{R}^*(M_1, M_2)$ , we have

$$\begin{aligned} R_1 &\geq c_1 R_1(\alpha^*, \beta^*) - c_2, \\ R_2 &\geq c_1 R_2(\alpha^*, \beta^*) - c_2. \end{aligned}$$

Since  $\mathcal{R}_C(M_1, M_2)$  is precisely the set of tuples of the form  $(R_1^*(\alpha, \beta), R_2^*(\alpha, \beta))$  for some  $\alpha, \beta \in [0, 1]$ , this shows that  $\mathcal{R}^*(M_1, M_2) \subseteq c_1 \cdot \mathcal{R}_C(M_1, M_2) - c_2$ , completing the proof. ■

As mentioned in Section IV, the proof above shows a stronger result than claimed in the theorem statement. In particular, it shows that for any  $M_1$  and  $M_2$  there exists parameters  $\alpha^*$  and  $\beta^*$  such that both  $R_1(\alpha^*, \beta^*)$  and  $R_2(\alpha^*, \beta^*)$  are *simultaneously* approximately close to their minimum value. In other words, up to a constant additive and multiplicative gap, there is no tension between the rates over the first and second hops of the network for the optimal caching scheme.

## APPENDIX A RATES FOR THE GENERALIZED CACHING SCHEME

This appendix derives the rate expressions (10) and (11) in Section V-C for the two subsystems using the generalized caching scheme.

Recall that the first subsystem is concerned with caching and delivering the first  $\alpha$  fraction of each file. It includes the entire memory of each mirror and the first  $\beta$  fraction of each user cache. Let

$$\begin{aligned} F^1 &\triangleq \alpha F, \\ M_1^1 &\triangleq \frac{M_1 F}{F^1} = \frac{M_1}{\alpha}, \\ M_2^1 &\triangleq \frac{\beta M_2 F}{F^1} = \frac{\beta M_2}{\alpha} \end{aligned}$$

denote the equivalent file size, as well as mirror memory and user cache memory, normalized by the equivalent file size, for this subsystem. From (8), the rates  $R_1^1, R_2^1$  (normalized by the file size  $F$ ) required by caching scheme A on this subsystem are given by

$$\begin{aligned} R_1^1 &= \alpha K_2 r\left(\frac{M_1^1}{N}, K_1\right) = \alpha K_2 r\left(\frac{M_1}{\alpha N}, K_1\right), \\ R_2^1 &= \alpha r\left(\frac{M_2^1}{N}, K_2\right) = \alpha r\left(\frac{\beta M_2}{\alpha N}, K_2\right). \end{aligned}$$



The second subsystem is concerned with caching and delivering the second  $1 - \alpha$  fraction of each file. It only uses the memory in the second  $1 - \beta$  fraction of each user cache. Let

$$F^2 \triangleq (1 - \alpha)F,$$

$$M_2^2 \triangleq \frac{(1 - \beta)M_2 F}{F^2} = \frac{(1 - \beta)M_2}{(1 - \alpha)}$$

denote the equivalent file size and user cache memory, normalized by the equivalent file size, for this subsystem. From (9), the rates  $R_1^2, R_2^2$  (again normalized by the file size  $F$ ) required by caching scheme B on this subsystem are given by

$$R_1^2 = (1 - \alpha)r\left(\frac{M_2^2}{N}, K_1 K_2\right) = (1 - \alpha)r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1 K_2\right),$$

$$R_2^2 = (1 - \alpha)r\left(\frac{M_2^2}{N}, K_2\right) = (1 - \alpha)r\left(\frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2\right).$$

## APPENDIX B LOWER BOUNDS

Given any  $M_1, M_2$ , we want to establish lower bounds on the rates  $R_1, R_2$  for the tuple  $(M_1, M_2, R_1, R_2)$  to be achievable. Our lower bounds are similar to the one proposed in [7] for single-layer caching networks.

Assume the tuple  $(M_1, M_2, R_1, R_2)$  is feasible and consider the shared communication link between the server and the mirrors. Fix  $s_1 \in \{1, 2, \dots, K_1\}$  and  $s_2 \in \{1, 2, \dots, K_2\}$ . Consider the set of  $s_1 \cdot s_2$  users  $(i, j)$  with  $i \in \{1, 2, \dots, s_1\}$  and  $j \in \{1, 2, \dots, s_2\}$ . Consider a request matrix  $\mathbf{D}$  with user  $(i, j)$  requesting  $d_{i,j} = (i - 1)s_2 + j$ . Since the tuple  $(M_1, M_2, R_1, R_2)$  is feasible, each user  $(i, j)$  can recover its requested file from the transmission from the server of rate  $R_1$  along with the contents of mirror  $i$  of size  $M_1$  and cache  $(i, j)$  of size  $M_2$ .

Now, consider a different request matrix  $\mathbf{D}$  in which user  $(i, j)$  requests  $d_{i,j} = s_1 s_2 + (i - 1)s_2 + j$ . Again from the server transmission of rate  $R_1$  and the two cache memories of sizes  $M_1$  and  $M_2$  each user  $(i, j)$  can recover its requested file. Note that, while the transmission of the server can depend on the request matrix, the contents of the caches do not.

Repeat the same argument for a total of  $\lfloor N/(s_1 s_2) \rfloor$  request matrices. Then we have the following cut-set bound [13]:

$$\left\lfloor \frac{N}{s_1 s_2} \right\rfloor R_1 + s_1 M_1 + s_1 s_2 M_2 \geq \left\lfloor \frac{N}{s_1 s_2} \right\rfloor s_1 s_2. \quad (21)$$

On the left-hand side of (21), the first term corresponds to the  $\lfloor N/(s_1 s_2) \rfloor$  transmissions from the server, one for each request matrix, of rate  $R_1$  each; the second term corresponds to the  $s_1$  mirror memories; and the third term corresponds to the  $s_1 s_2$  user memories. The right-hand side of (21) corresponds to the  $s_1 s_2$  different files that are reconstructed by the users for each of the  $\lfloor N/(s_1 s_2) \rfloor$  request matrices. (21) can be rewritten as

$$\begin{aligned} R_1 &\geq s_1 s_2 - \frac{s_1 M_1 + s_1 s_2 M_2}{\lfloor N/(s_1 s_2) \rfloor} \\ &\geq s_1 s_2 - \frac{s_1 M_1}{N/(s_1 s_2) - 1} - \frac{s_1 s_2 M_2}{N/(s_1 s_2) - 1} \\ &= s_1 s_2 \left( 1 - \frac{s_1 M_1 + s_1 s_2 M_2}{N - s_1 s_2} \right). \end{aligned} \quad (22)$$

We can modify the above argument slightly to get an alternate lower bound on the rate  $R_1$ . Instead of  $\lfloor N/(s_1 s_2) \rfloor$  transmissions, we will use  $\lceil N/(s_1 s_2) \rceil$  transmissions in (21) to get

$$\left\lceil \frac{N}{s_1 s_2} \right\rceil R_1 + s_1 M_1 + s_1 s_2 M_2 \geq N,$$

or, equivalently,

$$\begin{aligned}
 R_1 &\geq \frac{N - s_1 M_1 - s_1 s_2 M_2}{\lceil N/(s_1 s_2) \rceil} \\
 &\geq \frac{N - s_1 M_1 - s_1 s_2 M_2}{N/(s_1 s_2) + 1} \\
 &= \frac{s_1 s_2 (N - s_1 M_1 - s_1 s_2 M_2)}{N + s_1 s_2}.
 \end{aligned} \tag{23}$$

Since the inequalities (22) and (23) hold true for any choice of  $s_1 \in \{1, 2, \dots, K_1\}$  and  $s_2 \in \{1, 2, \dots, K_2\}$ , we have the following lower bound on the rate  $R_1$  for the tuple  $(M_1, M_2, R_1, R_2)$  to be feasible:

$$\begin{aligned}
 R_1 &\geq \max_{\substack{s_1 \in \{1, 2, \dots, K_1\} \\ s_2 \in \{1, 2, \dots, K_2\}}} \max \left\{ s_1 s_2 \left( 1 - \frac{s_1 M_1 + s_1 s_2 M_2}{N - s_1 s_2} \right), \frac{s_1 s_2 (N - s_1 M_1 - s_1 s_2 M_2)}{N + s_1 s_2} \right\} \\
 &\triangleq R_1^{\text{lb}}(M_1, M_2).
 \end{aligned} \tag{24}$$

#### A. Rate $R_2$

Assume the tuple  $(M_1, M_2, R_1, R_2)$  is feasible and consider the link between mirror 1 and its attached users. Let  $t \in \{1, 2, \dots, K_2\}$ . Consider the set of  $t$  users  $(1, j)$  with  $j \in \{1, 2, \dots, t\}$ . Consider a request matrix  $\mathbf{D}$  with user  $(1, j)$  requesting  $d_{1,j} = j$ . Since the tuple  $(M_1, M_2, R_1, R_2)$  is feasible, each user  $(1, j)$  can recover its requested file from the message transmitted by mirror 1 of rate  $R_2$  and the contents of its cache of size  $M_2$ .

Now, consider a different request matrix  $\mathbf{D}$  in which user  $(1, j)$  requests  $d_{1,j} = t + j$ . Again from the mirror transmission of rate  $R_2$  and its cache of size  $M_2$  each user  $(1, j)$  can recover its requested file. Note that, while the transmission of the mirror can depend on the request matrix, the contents of the caches do not.

Repeat the same argument for a total of  $\lfloor N/t \rfloor$  request matrices. Then we have the following cut-set bound [13]:

$$\left\lfloor \frac{N}{t} \right\rfloor R_2 + t M_2 \geq \left\lfloor \frac{N}{t} \right\rfloor t,$$

or, equivalently,

$$R_2 \geq t - \frac{t M_2}{\lfloor N/t \rfloor}.$$

Since this inequality holds true for any choice of  $t \in \{1, 2, \dots, K_2\}$ , we have the following lower bound on the rate  $R_2$  for the tuple  $(M_1, M_2, R_1, R_2)$  to be feasible:

$$\begin{aligned}
 R_2 &\geq \max_{t \in \{1, 2, \dots, K_2\}} t - \frac{t M_2}{\lfloor N/t \rfloor} \\
 &\geq \max_{t \in \{1, 2, \dots, K_2\}} t - \frac{t M_2}{\frac{N}{t} - 1} \\
 &= \max_{t \in \{1, 2, \dots, K_2\}} t - \frac{t^2 M_2}{N - t} \\
 &\triangleq R_2^{\text{lb}}(M_1, M_2).
 \end{aligned} \tag{25}$$

### APPENDIX C

GAP BETWEEN ACHIEVABLE RATE  $R_1(\alpha^*, \beta^*)$  AND LOWER BOUND  $R_1^{\text{lb}}(M_1, M_2)$

We prove that the rate  $R_1(\alpha^*, \beta^*)$  over the first hop, for the generalized caching scheme, as described in (18a) is within a constant additive and multiplicative gap of the minimum feasible rate  $R_1$  for all values of  $M_1, M_2$ . Recall from (6) and Fig. 3 that we use different parameters  $(\alpha^*, \beta^*)$  for the generalized caching scheme in the three different regimes of  $(M_1, M_2)$ , regimes I, II, and III. To prove the result, we will consider each of these regimes of  $(M_1, M_2)$  in sequence, and bound the gap between the achievable rate  $R_1(\alpha^*, \beta^*)$  and the corresponding lower bound  $R_1^{\text{lb}}(M_1, M_2)$ , as derived in Appendix B. Henceforth, we focus on the case where  $K_1, K_2 \geq 4$ . For  $K_1 \leq 3$  ( $K_2 \leq 3$ ), it is easy to see that the optimal rate is within the constant factor 3 of the rate of the network with  $K_1 = 1$  ( $K_2 = 1$ ). The optimum rate for  $K_1 = 1$  ( $K_2 = 1$ ) can be characterized easily following the results of [7].

We begin with regime I.

Regime I:  $M_1 + M_2 K_2 \geq N$ ,  $0 \leq M_1 < \frac{N}{4}$

For this regime, recall from (18a) that the achievable rate  $R_1(\alpha^*, \beta^*)$  is upper bounded as

$$R_1(\alpha^*, \beta^*) \leq \min \left\{ K_1 K_2, \frac{N}{M_2} \right\}. \quad (26)$$

On the other hand, recall the following lower bound on the rate  $R_1$  from (24):

$$R_1^{\text{lb}}(M_1, M_2) \geq \max_{\substack{s_1 \in \{1, 2, \dots, K_1\} \\ s_2 \in \{1, 2, \dots, K_2\}}} \frac{s_1 s_2 (N - s_1 M_1 - s_1 s_2 M_2)}{N + s_1 s_2} \quad (27)$$

For characterizing the gap between the achievable rate and the lower bound, we further divide this regime into three subregimes as follows:

$$\text{I.A)} \quad 0 \leq M_1 < \frac{N}{2K_1}, \quad \frac{3N}{4K_2} \leq M_2 < \frac{N}{4},$$

$$\text{I.B)} \quad \frac{N}{2K_1} \leq M_1 < \frac{N}{4}, \quad \frac{3N}{4K_2} \leq M_2 < \frac{N}{4},$$

$$\text{I.C)} \quad 0 \leq M_1 < \frac{N}{4}, \quad \frac{N}{4} \leq M_2 \leq N.$$

The subregimes above only consider  $M_2 \geq 3N/(4K_2)$  since for regime I, we have  $M_1 + M_2 K_2 \geq N$  and  $M_1 < N/4$ , and thus  $M_2 \geq (N - M_1)/K_2 \geq 3N/(4K_2)$ . We now consider the three subregimes one by one.

$$\text{I.A)} \quad 0 \leq M_1 < \frac{N}{2K_1}, \quad \frac{3N}{4K_2} \leq M_2 < \frac{N}{4}: \text{ Let}$$

$$\begin{aligned} s_1 &= 1, \\ s_2 &= \left\lfloor \frac{N}{2M_2} \right\rfloor \end{aligned}$$

in the lower bound in (27). Using  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , we can confirm that this is a valid choice since

$$1 \leq \frac{N}{4M_2} \leq \left\lfloor \frac{N}{2M_2} \right\rfloor \leq \frac{N}{2M_2} \leq \frac{2K_2}{3}. \quad (28)$$

Then, by evaluating (27) we have

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \frac{\left\lfloor \frac{N}{2M_2} \right\rfloor \left( N - M_1 - \left\lfloor \frac{N}{2M_2} \right\rfloor M_2 \right)}{N + \left\lfloor \frac{N}{2M_2} \right\rfloor} \\
&\stackrel{(a)}{\geq} \frac{\frac{N}{4M_2} \left( N - \frac{N}{2K_1} - \frac{N}{2M_2} M_2 \right)}{N + \frac{N}{2M_2}} \\
&\stackrel{(b)}{\geq} \frac{6N}{4 \cdot 7 \cdot M_2} \left( 1 - \frac{1}{2K_1} - \frac{1}{2} \right) \\
&\stackrel{(c)}{\geq} \frac{3N}{14M_2} \left( \frac{1}{2} - \frac{1}{8} \right) \\
&\geq \frac{N}{13M_2} \\
&\geq \frac{1}{13} \min \left\{ K_1 K_2, \frac{N}{M_2} \right\}
\end{aligned}$$

where (a) follows since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ ; (b) follows since

$$\frac{N}{2M_2} \leq \frac{2K_2}{3} = \frac{2K_1 K_2}{3K_1} \leq \frac{2N}{3K_1} \leq \frac{N}{6}$$

using (28),  $N \geq K_1 K_2$ , and  $K_1 \geq 4$ ; and (c) follows since we have  $K_1 \geq 4$ . Combining with (26), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{13} R_1(\alpha^*, \beta^*). \quad (29)$$

I.B)  $\frac{N}{2K_1} \leq M_1 < \frac{N}{4}$ ,  $\frac{3N}{4K_2} \leq M_2 < \frac{N}{4}$ : Let

$$(s_1, s_2) = \begin{cases} \left( \left\lfloor \frac{N}{4M_1} \right\rfloor, \left\lfloor \frac{M_1}{M_2} \right\rfloor \right) & \text{if } M_1 \geq M_2, \\ \left( \left\lfloor \frac{N}{4M_2} \right\rfloor, 1 \right) & \text{otherwise,} \end{cases}$$

in (27). This is a valid choice since for  $M_1 \geq M_2$ , we have

$$\begin{aligned}
1 &= \left\lfloor \frac{N}{4 \cdot N/4} \right\rfloor \leq \left\lfloor \frac{N}{4M_1} \right\rfloor \leq \frac{N}{4M_1} \leq \frac{K_1}{2}, \\
1 &\leq \left\lfloor \frac{M_1}{M_2} \right\rfloor \leq \frac{M_1}{M_2} \leq \frac{N/4}{3N/(4K_2)} = \frac{K_2}{3},
\end{aligned}$$

and for  $M_1 < M_2$ , we have

$$1 = \left\lfloor \frac{N}{4 \cdot N/4} \right\rfloor \leq \left\lfloor \frac{N}{4M_2} \right\rfloor \leq \left\lfloor \frac{N}{4M_1} \right\rfloor \leq \frac{N}{4M_1} \leq \frac{K_1}{2}.$$

Note that  $s_1 \leq N/(4M_1)$  and  $s_1 s_2 \leq N/(4M_2)$ . Further, since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , we have  $s_1 s_2 \geq N/(16M_2)$ . Finally, substituting  $s_1, s_2$  in (27), we obtain

$$\begin{aligned} R_1^{\text{lb}}(M_1, M_2) &\geq \frac{\frac{N}{16M_2} \left( N - \frac{N}{4M_1} \cdot M_1 - \frac{N}{4M_2} \cdot M_2 \right)}{N + \frac{N}{4M_2}} \\ &\stackrel{(a)}{\geq} \frac{\frac{N}{16M_2} \cdot \frac{N}{2}}{N + \frac{N}{12}} \\ &\geq \frac{N}{35M_2} \\ &\geq \frac{1}{35} \min \left\{ K_1 K_2, \frac{N}{M_2} \right\}. \end{aligned}$$

where (a) follows from

$$\frac{N}{4M_2} \leq \frac{K_2}{3} = \frac{K_1 K_2}{3K_1} \leq \frac{N}{3K_1} \leq \frac{N}{12}$$

using  $N \geq K_1 K_2$  and  $K_1 \geq 4$ . Combining with (26), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{35} R_1(\alpha^*, \beta^*). \quad (30)$$

I.C)  $0 \leq M_1 < \frac{N}{4}$ ,  $\frac{N}{4} \leq M_2 \leq N$ : We trivially have

$$R_1^{\text{lb}}(M_1, M_2) \geq 0 \geq \frac{N}{M_2} - 4 \geq \min \left\{ K_1 K_2, \frac{N}{M_2} \right\} - 4.$$

Combined with (26), this yields

$$R_1^{\text{lb}}(M_1, M_2) \geq R_1(\alpha^*, \beta^*) - 4. \quad (31)$$

Sections I.A, I.B, and I.C cover all the cases in regime I. Combining (29), (30), and (31), it follows that the achievable rate  $R_1(\alpha^*, \beta^*)$  and the lower bound  $R_1^{\text{lb}}(M_1, M_2)$  are within a constant multiplicative and additive gap for this regime.

Regime II:  $M_1 + M_2 K_2 < N$

For this regime, recall from (18a) that the achievable rate  $R_1(\alpha^*, \beta^*)$  is upper bounded as

$$R_1(\alpha^*, \beta^*) \leq \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\}. \quad (32)$$

On the other hand, (24) provides the following lower bound on the rate  $R_1$ :

$$R_1^{\text{lb}}(M_1, M_2) \geq \max_{\substack{s_1 \in \{1, 2, \dots, K_1\} \\ s_2 \in \{1, 2, \dots, K_2\}}} s_1 s_2 \left( 1 - \frac{s_1 M_1 + s_1 s_2 M_2}{N - s_1 s_2} \right). \quad (33)$$

For characterizing the gap between the achievable rate and the lower bounds, we further divide this regime into the following subregimes:

$$\text{II.A)} \quad 0 \leq M_1 < \frac{2N}{K_1}, \quad 0 \leq M_2 < \frac{2N}{K_1 K_2},$$

$$\text{II.B)} \quad 0 \leq M_1 < \frac{2N}{K_1}, \quad \frac{2N}{K_1 K_2} \leq M_2 < \frac{N}{2K_2},$$

$$\text{II.C)} \quad 0 \leq M_1 < \frac{N}{2K_1}, \quad \frac{N}{2K_2} \leq M_2 < \frac{N}{4},$$

$$\text{II.D)} \quad \frac{N}{2K_1} \leq M_1 < \frac{N}{4}, \quad 0 \leq M_2 < \frac{N}{4K_2},$$

$$\text{II.E)} \quad \frac{N}{2K_1} \leq M_1 < \frac{N}{4}, \quad \frac{N}{4K_2} \leq M_2 < \frac{N}{4},$$

$$\text{II.F)} \quad \frac{N}{4} \leq M_1 \leq N, \quad 0 \leq M_2 < \frac{N - M_1}{2K_2},$$

$$\text{II.G)} \quad \frac{N}{4} \leq M_1 \leq N, \quad \frac{N - M_1}{2K_2} \leq M_2 < \frac{N - M_1}{K_2}.$$

The subregimes above only consider  $M_2 < N/4$  since from the definition of regime II, we have

$$M_2 < \frac{N - M_1}{K_2} \leq \frac{N}{K_2} \leq \frac{N}{4}$$

using  $K_2 \geq 4$ . We now consider the different subregimes one by one.

$$\text{II.A)} \quad 0 \leq M_1 < \frac{2N}{K_1}, \quad 0 \leq M_2 < \frac{2N}{K_1 K_2} : \text{ Let}$$

$$s_1 = \left\lfloor \frac{K_1}{4} \right\rfloor, \\ s_2 = \left\lfloor \frac{K_2}{2} \right\rfloor$$

in the lower bound (27). This is a valid choice since  $K_1, K_2 \geq 4$ , and thus  $\lfloor K_1/4 \rfloor, \lfloor K_2/2 \rfloor \geq 1$ . Evaluating (27), we obtain

$$\begin{aligned} R_1^{\text{lb}}(M_1, M_2) &\geq \frac{\lfloor \frac{K_1}{4} \rfloor \lfloor \frac{K_2}{2} \rfloor (N - \lfloor \frac{K_1}{4} \rfloor M_1 - \lfloor \frac{K_1}{4} \rfloor \lfloor \frac{K_2}{2} \rfloor M_2)}{N + \lfloor \frac{K_1}{4} \rfloor \lfloor \frac{K_2}{2} \rfloor} \\ &\stackrel{(a)}{\geq} \frac{\frac{K_1 K_2}{32} (N - \frac{M_1 K_1}{4} - \frac{M_2 K_1 K_2}{8})}{N + \frac{K_1 K_2}{8}} \\ &\stackrel{(b)}{\geq} \frac{\frac{K_1 K_2}{32} (N - \frac{N}{2} - \frac{N}{4})}{N + \frac{N}{8}} \\ &= \frac{K_1 K_2}{144} \\ &\geq \frac{1}{144} \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\} \end{aligned}$$

where (a) follows since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ ; and (b) follows from  $M_1 < 2N/K_1$ ,  $M_2 \leq 2N/(K_1 K_2)$ , and  $N \geq K_1 K_2$ . Combining with (32), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{144} R_1(\alpha^*, \beta^*). \quad (34)$$

II.B)  $0 \leq M_1 < \frac{2N}{K_1}$ ,  $\frac{2N}{K_1 K_2} \leq M_2 < \frac{N}{2K_2}$ : Let

$$s_1 = \left\lfloor \frac{N}{2M_2 K_2} \right\rfloor, \\ s_2 = \left\lfloor \frac{K_2}{4} \right\rfloor$$

in (33). Note that this is a valid choice since

$$1 \leq \left\lfloor \frac{N}{2M_2 K_2} \right\rfloor \leq \frac{N}{2M_2 K_2} \leq \frac{K_1}{4}.$$

Further, we have  $K_2 \geq 4$  and thus,  $\lfloor K_2/4 \rfloor \geq 1$ . Substituting  $s_1, s_2$  in (33), we have

$$\begin{aligned} R_1^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{N}{2M_2 K_2} \right\rfloor \left\lfloor \frac{K_2}{4} \right\rfloor \left( 1 - \frac{\left\lfloor \frac{N}{2M_2 K_2} \right\rfloor M_1 + \left\lfloor \frac{N}{2M_2 K_2} \right\rfloor \left\lfloor \frac{K_2}{4} \right\rfloor M_2}{N - \left\lfloor \frac{N}{2M_2 K_2} \right\rfloor \left\lfloor \frac{K_2}{4} \right\rfloor} \right) \\ &\stackrel{(a)}{\geq} \frac{N}{32M_2} \left( 1 - \frac{\frac{NM_1}{2M_2 K_2} + \frac{N}{8}}{N - \frac{N}{8M_2}} \right) \\ &= \frac{N}{\left(1 - \frac{1}{8M_2}\right) 32M_2} \left( 1 - \frac{1}{8M_2} - \frac{M_1}{2M_2 K_2} - \frac{1}{8} \right) \\ &\stackrel{(b)}{\geq} \frac{N}{32M_2} \left( 1 - \frac{1}{16} - \frac{1}{2} - \frac{1}{8} \right) \\ &\geq \frac{N}{103M_2} \\ &\geq \frac{1}{103} \min \left\{ K_1 K_2, \frac{N}{M_2} \right\} \\ &\geq \frac{1}{103} \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1)K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{NK_2 - M_1}{M_1 + M_2 K_2} \right\} \end{aligned}$$

where (a) follows from  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$  and from  $N - N/(8M_2) > 0$  using

$$\frac{N}{8M_2} \leq \frac{K_1 K_2}{16} \leq \frac{N}{16},$$

and (b) follows from  $N/(8M_2) \leq N/16$  as shown above and from  $M_1 < 2N/K_1$ ,  $M_2 \geq 2N/(K_1 K_2)$ . Combined with (32), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{103} R_1(\alpha^*, \beta^*). \quad (35)$$

II.C)  $0 \leq M_1 < \frac{2N}{K_1}$ ,  $\frac{N}{2K_2} \leq M_2 < \frac{N}{4}$ : Let

$$s_1 = 1,$$

$$s_2 = \left\lfloor \frac{N}{4M_2} \right\rfloor$$

in the lower bound in (33). Using  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , we can confirm that this is a valid choice since

$$1 = \left\lfloor \frac{N}{4 \cdot N/4} \right\rfloor \leq \left\lfloor \frac{N}{4M_2} \right\rfloor \leq \frac{N}{4M_2} \leq \frac{K_2}{2}. \quad (36)$$

Evaluating (33), we obtain

$$\begin{aligned} R_1^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{N}{4M_2} \right\rfloor \left( 1 - \frac{M_1 + \left\lfloor \frac{N}{4M_2} \right\rfloor M_2}{N - \left\lfloor \frac{N}{4M_2} \right\rfloor} \right) \\ &\stackrel{(a)}{\geq} \frac{N}{8M_2} \left( 1 - \frac{M_1 + \frac{N}{4}}{N - \frac{N}{4M_2}} \right) \\ &= \frac{1}{\left(1 - \frac{1}{4M_2}\right)} \frac{N}{8M_2} \left( 1 - \frac{1}{4M_2} - \frac{M_1}{N} - \frac{1}{4} \right) \\ &\stackrel{(b)}{\geq} \frac{N}{8M_2} \left( 1 - \frac{1}{8} - \frac{1}{2} - \frac{1}{4} \right) \\ &\geq \frac{N}{64M_2} \\ &\geq \frac{1}{64} \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\} \end{aligned}$$

where (a) follows from  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , and from  $N - N/(4M_2) > 0$  since

$$\frac{N}{4M_2} \leq \frac{K_2}{2} = \frac{K_1 K_2}{2K_1} \leq \frac{N}{8}$$

using (36),  $N \geq K_1 K_2$  and  $K_1 \geq 4$ ; and (b) follows from  $N/(4M_2) \leq N/8$  as shown above and  $M_1 < 2N/K_1 \leq N/2$  using  $K_1 \geq 4$ . Combining with (26), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{64} R_1(\alpha^*, \beta^*). \quad (37)$$

II.D)  $\frac{2N}{K_1} \leq M_1 < \frac{N}{4}$ ,  $0 \leq M_2 < \frac{N}{4K_2}$ : Let

$$s_1 = \left\lfloor \frac{N}{2(M_1 + M_2 K_2)} \right\rfloor,$$

$$s_2 = K_2$$

in (33). Note that this is a valid choice since

$$1 = \left\lfloor \frac{N}{2(N/4 + N/4)} \right\rfloor \leq \left\lfloor \frac{N}{2(M_1 + M_2 K_2)} \right\rfloor \leq \frac{N}{2(M_1 + M_2 K_2)} \leq \frac{N}{2M_1} \leq K_1.$$



Substituting  $s_1, s_2$  in (33), we obtain

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{N}{2(M_1 + M_2 K_2)} \right\rfloor K_2 \left( 1 - \frac{\left\lfloor \frac{N}{2(M_1 + M_2 K_2)} \right\rfloor (M_1 + M_2 K_2)}{N - \left\lfloor \frac{N}{2(M_1 + M_2 K_2)} \right\rfloor K_2} \right) \\
&\stackrel{(a)}{\geq} \frac{N}{4(M_1 + M_2 K_2)} K_2 \left( 1 - \frac{\frac{N}{2(M_1 + M_2 K_2)} (M_1 + M_2 K_2)}{N - \frac{N K_2}{2M_1}} \right) \\
&\stackrel{(b)}{\geq} \frac{N K_2}{4(M_1 + M_2 K_2)} \left( 1 - \frac{\frac{N}{2}}{N - \frac{N}{4}} \right) \\
&= \frac{N K_2}{12(M_1 + M_2 K_2)} \\
&\geq \frac{1}{12} \min \left\{ K_1 K_2, \frac{N K_2}{M_1 + M_2 K_2} \right\} \\
&\geq \frac{1}{12} \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\}
\end{aligned}$$

where (a) follows since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$  and  $N - N K_2/(2M_1) > 0$  since

$$\frac{N K_2}{2M_1} \leq \frac{N K_2}{2} \cdot \frac{K_1}{2N} = \frac{K_1 K_2}{4} \leq \frac{N}{4},$$

and (b) follows from  $N K_2/(2M_1) \leq N/4$  as shown above. Combining with (32), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{12} R_1(\alpha^*, \beta^*). \quad (38)$$

II.E)  $\frac{2N}{K_1} \leq M_1 < \frac{N}{4}$ ,  $\frac{N}{4K_2} \leq M_2 < \frac{N}{4}$ : Let

$$(s_1, s_2) = \begin{cases} \left( \left\lfloor \frac{N}{4M_1} \right\rfloor, \left\lfloor \frac{M_1}{M_2} \right\rfloor \right) & \text{if } M_1 \geq M_2, \\ \left( \left\lfloor \frac{N}{4M_2} \right\rfloor, 1 \right) & \text{otherwise,} \end{cases}$$

in (33). This is a valid choice since for  $M_1 \geq M_2$ , we have

$$\begin{aligned}
1 &= \left\lfloor \frac{N}{4 \cdot N/4} \right\rfloor \leq \left\lfloor \frac{N}{4M_1} \right\rfloor \leq \frac{N}{4M_1} \leq \frac{K_1}{8}, \\
1 &\leq \left\lfloor \frac{M_1}{M_2} \right\rfloor \leq \frac{M_1}{M_2} \leq \frac{N/4}{N/(4K_2)} = K_2,
\end{aligned}$$

and for  $M_1 < M_2$ , we have

$$4 = \left\lfloor \frac{N}{4 \cdot N/4} \right\rfloor \leq \left\lfloor \frac{N}{4M_2} \right\rfloor \leq \left\lfloor \frac{N}{4M_1} \right\rfloor \leq \frac{N}{4M_1} \leq \frac{K_1}{8}.$$

Note that  $s_1 \leq N/(4M_1)$  and  $s_1 s_2 \leq N/(4M_2)$ . Further, since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , we have  $s_1 s_2 \geq N/(16M_2)$ . Also, note that  $N - N/(4M_2) > 0$  since

$$\frac{N}{4M_2} \leq K_2 = \frac{K_1 K_2}{K_1} \leq \frac{K_1 K_2}{4} \leq \frac{N}{4},$$

using  $N \geq K_1 K_2$  and  $K_1 \geq 4$ . Finally, substituting  $s_1, s_2$  in (33), we obtain

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \frac{N}{16M_2} \left( 1 - \frac{\frac{N}{4M_1} \cdot M_1 + \frac{N}{4M_2} \cdot M_2}{N - \frac{N}{4M_2}} \right) \\
&\geq \frac{N}{16M_2} \left( 1 - \frac{\frac{N}{2}}{N - \frac{N}{4}} \right) \\
&= \frac{N}{48M_2} \\
&\geq \frac{1}{48} \min \left\{ K_1 K_2, \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{(N - M_1) K_2}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \right\}.
\end{aligned}$$

Combining with (32), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{48} R_1(\alpha^*, \beta^*). \quad (39)$$

II.F)  $\frac{N}{4} \leq M_1 \leq N$ ,  $0 \leq M_2 < \frac{N - M_1}{2K_2}$ : Substituting  $s_1 = 1, s_2 = K_2$  in the lower bound (27), we obtain

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \frac{K_2(N - M_1 - M_2 K_2)}{N + K_2} \\
&\stackrel{(a)}{\geq} \frac{K_2(N - M_1 - (N - M_1)/2)}{N + \frac{N}{4}} \\
&= \frac{2K_2(N - M_1)}{5N}
\end{aligned} \quad (40)$$

where (a) follows since

$$K_2 = \frac{K_1 K_2}{K_1} \leq \frac{K_1 K_2}{4} \leq \frac{N}{4}$$

using  $N \geq K_1 K_2$  and  $K_1 \geq 4$ . On the other hand, from (32) we obtain

$$\begin{aligned}
R_1(\alpha^*, \beta^*) &\leq \frac{M_1}{M_1 + M_2 K_2} \cdot \frac{K_2(N - M_1)}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2 - M_1}{M_1 + M_2 K_2} \\
&\leq \frac{K_2(N - M_1)}{M_1 + M_2 K_2} + \frac{M_2 K_2}{M_1 + M_2 K_2} \cdot \frac{N K_2}{M_1} \\
&\stackrel{(a)}{\leq} \frac{K_2(N - M_1)}{M_1 + M_2 K_2} + \frac{(N - M_1)}{2} \cdot \frac{N}{M_1} \cdot \frac{K_2}{(M_1 + M_2 K_2)} \\
&= \frac{K_2(N - M_1)}{M_1 + M_2 K_2} \left( 1 + \frac{N}{2M_1} \right) \\
&\stackrel{(b)}{\leq} \frac{3K_2(N - M_1)}{M_1 + M_2 K_2}
\end{aligned}$$

where (a) follows since  $M_2 < (N - M_1)/(2K_2)$  for this case; and (b) follows since  $M_1 \geq N/4$ . Combining with (40), we obtain

$$\begin{aligned}
 R_1^{\text{lb}}(M_1, M_2) &\geq \frac{2K_2(N - M_1)}{5N} \\
 &= \frac{2}{5 \cdot 3} \cdot \frac{M_1 + M_2K_2}{N} \cdot \frac{3K_2(N - M_1)}{M_1 + M_2K_2} \\
 &\stackrel{(a)}{\geq} \frac{2}{15} \cdot \frac{N/4}{N} \cdot \frac{3K_2(N - M_1)}{M_1 + M_2K_2} \\
 &\geq \frac{1}{30} R_1(\alpha^*, \beta^*)
 \end{aligned} \tag{41}$$

where (a) follow since  $M_1 \geq N/4$ .

$$\text{II.G) } \frac{N}{4} \leq M_1 \leq N, \quad \frac{N - M_1}{2K_2} \leq M_2 < \frac{N - M_1}{K_2} : \text{ Let}$$

$$\begin{aligned}
 s_1 &= 1, \\
 s_2 &= \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor
 \end{aligned}$$

in (27). This is a valid choice since  $K_2 \geq 4$  and  $M_1 + M_2K_2 < N$ , so that

$$1 \leq \left\lfloor \frac{N - M_1}{K_2M_2} \right\rfloor \leq \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \leq \frac{N - M_1}{2M_2} \leq K_2.$$

Substituting  $s_1, s_2$  in (27), we obtain

$$\begin{aligned}
 R_1^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \frac{\left(N - M_1 - \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor M_2\right)}{N + \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor} \\
 &\stackrel{(a)}{\geq} \frac{N - M_1}{4M_2} \frac{\left(N - M_1 - \frac{N - M_1}{2M_2} M_2\right)}{N + \frac{N}{4}} \\
 &= \frac{N - M_1}{4M_2} \cdot \frac{2(N - M_1)}{5N} \\
 &= \frac{(N - M_1)^2}{10M_2N}
 \end{aligned} \tag{42}$$

where (a) follows from  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$ , and

$$\left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \leq \frac{N - M_1}{2M_2} \leq K_2 \leq \frac{N}{K_1} \leq \frac{N}{4}$$

using  $N \geq K_1K_2$  and  $K_1 \geq 4$ . On the other hand, from (32) we obtain

$$\begin{aligned}
 R_1(\alpha^*, \beta^*) &\leq \frac{M_1}{M_1 + M_2K_2} \cdot \frac{K_2(N - M_1)}{M_1 + M_2K_2} + \frac{M_2K_2}{M_1 + M_2K_2} \cdot \frac{NK_2 - M_1}{M_1 + M_2K_2} \\
 &\stackrel{(a)}{\leq} \frac{K_2(N - M_1)}{M_1 + M_2K_2} + \frac{N - M_1}{M_1 + M_2K_2} \cdot \frac{NK_2}{M_1 + M_2K_2} \\
 &= \frac{K_2(N - M_1)}{M_1 + M_2K_2} \left(1 + \frac{N}{M_1 + M_2K_2}\right) \\
 &\stackrel{(b)}{\leq} \frac{3K_2(N - M_1)}{M_1 + M_2K_2}
 \end{aligned}$$

where (a) follows since  $M_1 + M_2 K_2 < N$  for regime II and (b) follows since  $M_1 + M_2 K_2 \geq M_1 + (N - M_1)/2 \geq N/2$  for this case. Combining with (42), we have

$$\begin{aligned}
 R_1^{\text{lb}}(M_1, M_2) &\geq \frac{(N - M_1)^2}{10M_2 N} \\
 &= \frac{1}{30} \cdot \frac{3K_2(N - M_1)}{M_1 + M_2 K_2} \cdot \frac{N - M_1}{M_2 K_2} \cdot \frac{M_1 + M_2 K_2}{N} \\
 &\stackrel{(a)}{\geq} \frac{1}{30} \cdot \frac{3K_2(N - M_1)}{M_1 + M_2 K_2} \cdot 1 \cdot \frac{1}{2} \\
 &\geq \frac{1}{60} R_1(\alpha^*, \beta^*)
 \end{aligned} \tag{43}$$

where (a) follows since  $N/2 \leq M_1 + M_2 K_2 < N$  for this case.

Sections II.A - II.G cover all the cases in regime II. Combining (34), (35), (37), (38), (39), (41), and (43) shows that the achievable rate  $R_1(\alpha^*, \beta^*)$  and the lower bound  $R_1^{\text{lb}}(M_1, M_2)$  are within a constant multiplicative and additive gap in this regime.

Regime III:  $M_1 + M_2 K_2 \geq N$ ,  $\frac{N}{4} \leq M_1 \leq N$

For this regime, recall from (18a) that the achievable rate  $R_1(\alpha^*, \beta^*)$  is upper bounded as

$$R_1(\alpha^*, \beta^*) \leq \frac{4(N - M_1)^2}{3NM_2}. \tag{44}$$

To characterize the gap between the achievable rate and the lower bounds, we further divide regime III into the two subregimes

$$\text{III.A) } \frac{N}{4} \leq M_1 \leq N, \quad \frac{N - M_1}{K_2} \leq M_2 < \frac{N - M_1}{2},$$

$$\text{III.B) } \frac{N}{4} \leq M_1 \leq N, \quad \frac{N - M_1}{2} \leq M_2 \leq N.$$

We now consider the subregimes one by one.

$$\text{III.A) } \frac{N}{4} \leq M_1 \leq N, \quad \frac{N - M_1}{K_2} \leq M_2 < \frac{N - M_1}{2}: \text{ Let}$$

$$\begin{aligned}
 s_1 &= 1, \\
 s_2 &= \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor
 \end{aligned}$$

in the lower bound (27). This is a valid choice since

$$1 \leq \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \leq \frac{N - M_1}{2M_2} \leq \frac{K_2}{2}. \tag{45}$$

Substituting  $s_1, s_2$  in (27), we obtain

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \frac{\left( N - M_1 - \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor M_2 \right)}{N + \left\lfloor \frac{N - M_1}{2M_2} \right\rfloor} \\
&\stackrel{(a)}{\geq} \frac{N - M_1}{4M_2} \frac{\left( N - M_1 - \frac{N - M_1}{2M_2} M_2 \right)}{N + \frac{N}{8}} \\
&= \frac{(N - M_1)^2}{9M_2N}
\end{aligned}$$

where (a) follows from  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$  and

$$\left\lfloor \frac{N - M_1}{2M_2} \right\rfloor \leq \frac{K_2}{2} = \frac{K_1 K_2}{2K_1} \leq \frac{N}{8}.$$

using (45),  $N \geq K_1 K_2$ , and  $K_1 \geq 4$ . Combining with (44), we have

$$R_1^{\text{lb}}(M_1, M_2) \geq \frac{(N - M_1)^2}{9M_2N} = \frac{1}{12} \cdot \frac{4(N - M_1)^2}{3NM_2} \geq \frac{1}{12} R_1(\alpha^*, \beta^*). \quad (46)$$

III.B)  $\frac{N}{4} \leq M_1 \leq N$ ,  $\frac{N - M_1}{2} \leq M_2 \leq N$ : We trivially have

$$R_1^{\text{lb}}(M_1, M_2) \geq 0 = \frac{8}{3} - \frac{8}{3}.$$

Combining with (44) and using  $(N - M_1)/M_2 \leq 2$  for this case, we have

$$\begin{aligned}
R_1^{\text{lb}}(M_1, M_2) &\geq \frac{4}{3} \cdot 2 \cdot 1 - \frac{8}{3} \\
&\geq \frac{4}{3} \cdot \frac{N - M_1}{M_2} \cdot \frac{N - M_1}{N} - \frac{8}{3} \\
&= \frac{4(N - M_1)^2}{3NM_2} - \frac{8}{3} \\
&\geq R_1(\alpha^*, \beta^*) - \frac{8}{3}.
\end{aligned} \quad (47)$$

Sections III.A and III.B cover all the cases in regime III. Combining (46) and (47), it follows that the achievable rate  $R_1(\alpha^*, \beta^*)$  and the lower bound  $R_1^{\text{lb}}(M_1, M_2)$  are within a constant multiplicative and additive gap for regime III.

Regimes I, II, and III cover all possible values for  $(M_1, M_2)$ . For each regime, we have shown that the achievable rate  $R_1$  for the generalized caching scheme is within a constant additive and multiplicative gap of the minimum feasible rate. In particular, for any  $M_1, M_2$ , and any feasible rate pair  $(R_1, R_2) \in \mathcal{R}^*(M_1, M_2)$ , we have

$$R_1 \geq R_1^{\text{lb}}(M_1, M_2) \geq \frac{1}{144} R_1(\alpha^*, \beta^*) - 4.$$

## APPENDIX D

GAP BETWEEN ACHIEVABLE RATE  $R_2(\alpha^*, \beta^*)$  AND LOWER BOUND  $R_2^{\text{lb}}(M_1, M_2)$

We prove that the rate  $R_2(\alpha^*, \beta^*)$  for the generalized caching scheme, as described in (18b), is within a constant additive and multiplicative gap of the corresponding lower bound  $R_2^{\text{lb}}(M_1, M_2)$ , as derived in Appendix B, for all values of  $M_1, M_2$ . As before, we focus on the case where  $K_1, K_2 \geq 4$ . The case of  $K_1 < 4$  or  $K_2 < 4$  is easily analyzed using the results of [7].

From (18b), we have the following upper bound on the achievable rate  $R_2(\alpha^*, \beta^*)$  of the generalized caching scheme for any  $M_1, M_2$ :

$$R_2(\alpha^*, \beta^*) \leq 4 \min \left\{ K_2, \frac{N}{M_2} \right\}. \quad (48)$$

On the other hand, recall the following lower bound on the rate  $R_2$  from (25):

$$R_2^{\text{lb}}(M_1, M_2) = \max_{t \in \{1, 2, \dots, K_2\}} t - \frac{t^2 M_2}{N - t}. \quad (49)$$

To characterize the gap between the achievable rate and the lower bounds, we study two different cases.

- 1)  $0 \leq M_2 < \frac{N}{4}$
- 2)  $\frac{N}{4} \leq M_2 \leq N$ .

We now consider the two cases one by one.

- 1)  $0 \leq M_2 < \frac{N}{4}$ : Let

$$t = \left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor$$

in (49). This is a valid choice since  $K_2 \geq 4$ , and thus

$$1 \leq \left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor \leq \frac{K_2}{4}.$$

Substituting  $t$  in (49) yields

$$\begin{aligned} R_2^{\text{lb}}(M_1, M_2) &\geq \left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor - \left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor^2 \cdot \frac{M_2}{N - \left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor} \\ &\stackrel{(a)}{\geq} \frac{1}{8} \min \left\{ K_2, \frac{N}{M_2} \right\} - \frac{1}{16} \min \left\{ K_2, \frac{N}{M_2} \right\}^2 \cdot \frac{M_2}{N - \frac{N}{16}} \\ &= \frac{1}{8 \cdot 15} \min \left\{ K_2, \frac{N}{M_2} \right\} \left( 15 - 8 \cdot \min \left\{ K_2, \frac{N}{M_2} \right\} \cdot \frac{M_2}{N} \right) \\ &\geq \frac{7}{120} \min \left\{ K_2, \frac{N}{M_2} \right\} \end{aligned} \quad (50)$$

where (a) follows since  $\lfloor x \rfloor \geq x/2$  for any  $x \geq 1$  and

$$\left\lfloor \frac{1}{4} \min \left\{ K_2, \frac{N}{M_2} \right\} \right\rfloor \leq \frac{K_2}{4} = \frac{K_1 K_2}{4 K_1} \leq \frac{N}{16}$$

using  $N \geq K_1 K_2$  and  $K_1 \geq 4$ . Comparing (48) and (50), we have

$$R_2^{\text{lb}}(M_1, M_2) \geq \frac{7}{120} \min \left\{ K_2, \frac{N}{M_2} \right\} = \frac{7}{120 \cdot 4} \cdot 4 \min \left\{ K_2, \frac{N}{M_2} \right\} \geq \frac{1}{70} R_2(\alpha^*, \beta^*). \quad (51)$$

2)  $\frac{N}{4} \leq M_2 \leq N$ : We trivially have

$$R_2^{\text{lb}}(M_1, M_2) \geq 0 = 4 \frac{N}{M_2} - 4 \frac{N}{M_2}.$$

Combining with (48), we have

$$R_2^{\text{lb}}(M_1, M_2) \geq 4 \frac{N}{M_2} - 4 \frac{N}{M_2} \geq 4 \min \left\{ K_2, \frac{N}{M_2} \right\} - 4 \frac{N}{M_2} \geq R_2(\alpha^*, \beta^*) - 16. \quad (52)$$

Cases 1) and 2) cover all values of the memory sizes  $M_1, M_2$ . Combining (51), (52), it follows that the achievable rate  $R_2(\alpha^*, \beta^*)$  of the generalized caching scheme and the lower bound  $R_2^{\text{lb}}(M_1, M_2)$  are within a constant multiplicative and additive gap for all values of  $M_1, M_2$ . In particular, for any  $M_1, M_2$ , and any feasible rate pair  $(R_1, R_2) \in \mathcal{R}^*(M_1, M_2)$ , we have

$$R_2 \geq R_2^{\text{lb}}(M_1, M_2) \geq \frac{1}{70} R_2(\alpha^*, \beta^*) - 16.$$

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